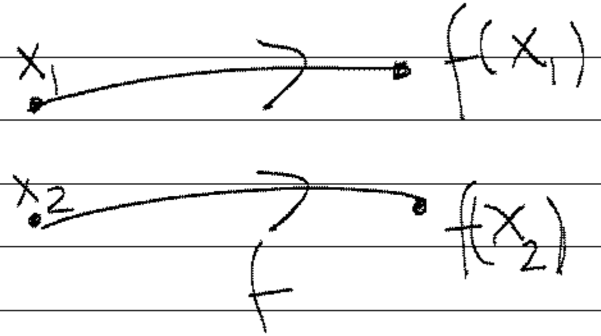
Note Title 1/13/2016

functions: $f: X \longrightarrow Y$

f is one to one if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$



f is onto if $\forall\, y \in Y$ $\exists\, x \in X$ such that $f(x) = y$

Ex: $f: \mathbb{R} \longrightarrow \mathbb{R}$ $f(x) = x^2$ is not onto because there is no x such that $f(x) = x^2 = -1$

$f: \mathbb{R} \to \mathbb{R}_{\geq 0} = \{\text{non-negative real numbers}\}$ $f(x) = x^2$ is onto

because if $y \in \mathbb{R}_{\geq 0}$ then $y = f(\sqrt{y})$

<u>Restriction of a function</u>: $f: X \to Y$ if $X' \subset X$

Let $g: X' \to Y$ be defined as $g(x) = f(x)$. We call g the restriction of f on X'. We usually denote g by f.

<u>Composition of functions</u>: $f: X \to Y$ $g: Y \to Z$

$g(f(x)) = (g \circ f)(x)$

<u>Def</u>: one to one = one-one

one-one and onto = one-one onto = one-one correspondence

Def $i_X : X \to X \qquad i_X(x) = x$

Def: $f : X \to Y$ one-one onto. Let $y \in Y$. $\exists!\, x$ such that $f(x) = y$ denote this x by $f^{-1}(y) = x$. This defines a function $f^{-1} : Y \to X$. This function is called the inverse of f.

Note: 1) $y = f(x) \implies f^{-1}(y) = x$ then $(f \circ f^{-1})(y) = f(f^{-1}(y)) = f(x) = y$

$f \circ f^{-1} = i_Y$

2) If $f(x) = y$, then $f^{-1}(y) = x$. Then $f^{-1}(y) = f^{-1}(f(x)) = x$

$f^{-1} \circ f = i_X$

3) $(f^{-1})^{-1} = f$

4) Given f, if there exists g such that $g \circ f = i$ $f \circ g = i$ then f is one-one onto

Definitions : 1) $f : X \to Y$. $X' \subset X$. then

$f(X') = \{f(x) : x \in X'\}$ (Note $f(X') \subset Y$). image of X'

$Y' \subset Y$ $f^{-1}(Y') = \{x \in X : f(x) \in Y'\}$ (Note $f^{-1}(Y') \subset X$)

pre-image of Y'

$f(X) = \text{range} = \text{image} \subset Y = \text{codomain}$

<u>Def</u>: 1) positive integers = natural numbers $= \{1, 2, 3, 4, \ldots\} = \mathbb{N}$

2) X is finite if $\exists\, n \in \mathbb{N}$ and $f: \{1, 2, \ldots, n\} \to X$ that is one-one onto

<u>Example</u> $X = \{1, 2, 7\}$ is finite $f: \{1, 2, 3\} \to \{1, 2, 7\}$

$f(1) = 1 \quad f(2) = 2 \quad f(3) = 7$

3) n as in 2) is called the number of elements in X

<u>Obs</u>: For the number of elements in a finite set to be well defined we need to show that if X is a set, and

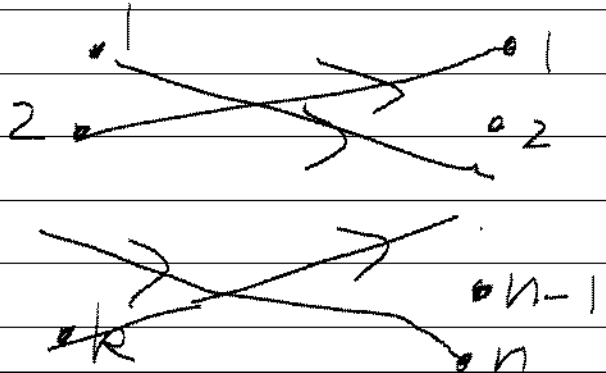
$f: \{1, \dots, n\} \longrightarrow X$
$g: \{1, \dots, k\} \longrightarrow X$ both one-one onto then $n = k$

Obs: $f: \{1, \dots, k\} \longrightarrow \{1, \dots, n\}$ one-one onto then $n = k$

proof: Induction on n

$\boxed{n=1}$ $f: \{1, \dots, k\} \longrightarrow \{1\}$ $f(1) = 1$ if $k \geq 2$ then $f(2) = 1$ but f is one-one, contradiction. Then $k < 2$ then $k = 1$

$\boxed{n \geq 2}$ let $l = f^{-1}(n)$



$I = \{1, \dots, k\} - \{f^{-1}(n)\}$

$f : I \longrightarrow \{1, \dots, n-1\}$ is well defined is one-one onto

$I = \{1, \dots, l-1\} \cup \{l+1, \dots, k\}$

$g : \{1, \dots, k-1\} \longrightarrow I$

$$
\begin{array}{ccc}
1 & \xrightarrow{g} & 1 \\
2 & \longrightarrow & 2 \\
\vdots & & \\
l-1 & \longrightarrow & l-1 \\
l & \longrightarrow & l+1 \\
\vdots & & \vdots \\
k-1 & \longrightarrow & k
\end{array}
$$

$$g(i) = \begin{cases} i & \text{if } 1 \le i \le l-1 \\ i+1 & \text{if } l \le i \le k \end{cases}$$

g is one-one onto

$(f \circ g) : \{1, \ldots, k-1\} \longrightarrow \{1, \ldots, n-1\}$ is one-one onto

Apply induction $k-1 = n-1 \Rightarrow k = n$.