Note Title: Real numbers 1/20/2016

F4) if $a \neq 0$ $\exists!\, x \in \mathbb{R}$ such that $x \cdot a = b$
$b \in \mathbb{R}$

multiply both sides by a^{-1} to get $x = b \cdot a^{-1}$

Obs: 1) If $a \neq 0$ and $a \cdot b = a \cdot c = 1$ then $c = b$

proof: $b = b \cdot 1 = b \cdot (a \cdot c) = (b \cdot a) \cdot c = 1 \cdot c = c$

2) If $a + b = 0$ and $a + c = 0 \implies b = c$

F5) $a \cdot 0 = 0 \quad \forall\, a \in \mathbb{R}$

$a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0$

add $-a \cdot 0$ on both sides to get $0 = a \cdot 0$

F6) $-(-a) = a$ <u>proof</u> $(-a) + -(-a) = 0$

$(-a) + a = 0$

thus $-(-a)$ and a are both solutions to $(-a) + X = 0$

Since $\exists!$ x that satisfies $(-a) + X = 0$ then $a = -(-a)$

F7) $(a^{-1})^{-1} = a$ if $a \neq 0$ because both a and $(a^{-1})^{-1}$ are the unique solution to $a^{-1}x = 1$

F8) $-(a+b) = (-a) + (-b) = -a-b$

again, $-(a+b)$ and $-a-b$ both solve $a+b+X = 0$

F9) $(ab)^{-1} = a^{-1}b^{-1}$ similar reasons

Notation $\frac{a}{b} = a/b = a b^{-1}$

Obs $\frac{ac}{bc} = ac(bc)^{-1} = a c b^{-1} c^{-1} = a c c^{-1} b^{-1} = a \cdot 1 \cdot b^{-1} = a b^{-1} = \frac{a}{b}$

$\frac{a}{b} \cdot \frac{c}{d} = a b^{-1} \cdot c d^{-1} = (ac)(bd)^{-1} = \frac{ac}{bd}$

$\frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd}$

F10) $-a = (-1)a$ because both sides solve $a + x = 0$

$a - a = 0$ ✓ $a + (-1)a = 1a + (-1)a = (1 + (-1))a = 0 \cdot a = 0$

$a(-b) = a\cdot(-1)\cdot b = (-1)\cdot a\cdot b = (-a)\cdot b = -ab$

$(-a)(-b) = \left(-(-a)\right) b = ab$

Section 2.2 ORDER

Property $\underline{\text{VI}}$ $\exists\ \mathbb{R}_+ \subset \mathbb{R}$ such that

(1) $a, b \in \mathbb{R}_+ \implies a+b \in \mathbb{R}_+ \quad a\cdot b \in \mathbb{R}_+$

(2) $a \in \mathbb{R} \Rightarrow$ one and only one of the following statements is true $\quad a \in \mathbb{R}_+$, or $a = 0$, or $-a \in \mathbb{R}_+$

<u>Name</u>: the elements in $\mathbb{R}_+$ are called positive numbers

If $-a \in \mathbb{R}_+$ we say a is negative

Notation: $a > b$ means $a - b \in \mathbb{R}_+$

$b < a$ means the same

$a \geq b$ means $a - b \in \mathbb{R}_+$ or $a - b = 0$

$b \leq a$ means the same

Obs: $a \in \mathbb{R}_+ \iff a > 0$

a is negative $\iff a < 0$

$\boxed{O1}$ $a, b \in \mathbb{R}$. Exactly one of the following is true

$a > b$, or $a = b$, or $a < b$

$\boxed{O2}$ $a > b$ and $b > c \implies a > c$

$$a-c = \underbrace{(a-b)}_{\in \mathbb{R}_+} + \underbrace{(b-c)}_{\in \mathbb{R}_+} \text{ thus } a-c \in \mathbb{R}_+$$

$\boxed{O_3}$ $a > b$ & $c \geq d$ then $a+c > b+d$

$$(a+c)-(b+d) = \underbrace{(a-b)}_{\in \mathbb{R}_+} + \underbrace{(c-d)}_{\mathbb{R}_+ \cup \{0\}} \Longrightarrow a+c-(b+d) \in \mathbb{R}_+$$

$\boxed{O4}$ $a > b > 0$ & $c \geq d > 0 \Rightarrow a \cdot c > b \cdot d$

$a \cdot c - b \cdot d$

$a-b > 0$ & $c > 0 \Rightarrow c \cdot (a-b) = a \cdot c - b \cdot c > 0$

$$c-d \geq 0 \ \& \ b>0 \Rightarrow b\cdot(c-d) = b\cdot c - b\cdot d \geq 0$$

$$a\cdot c - b\cdot d > 0$$

$$\boxed{a\cdot c > b\cdot d}$$