Note Title 1/25/2016

05 positive + positive = positive

negative + negative = negative

positive $\cdot$ positive = positive

positive $\cdot$ negative = negative

negative $\cdot$ negative = positive

06 $a \in \mathbb{R}$ $a^2 \geq 0$ and $a^2 = 0 \iff a = 0$

$1^2 = 1 > 0$

07 $a > 0$ $a \cdot (1/a) = 1 \implies 1/a > 0$

08 $\boxed{a > b > 0} \implies ab > 0 \implies (ab)^{-1} > 0 \implies$

$\Rightarrow \quad a(ab)^{-1} > b(ab)^{-1} \quad \Rightarrow \quad a a^{-1} b^{-1} > b a^{-1} b^{-1}$

$\Rightarrow \quad \boxed{b^{-1} > a^{-1} > 0}$

Natural numbers

$1, \quad 2 = 1+1, \quad 3 = 1+2, \quad 4 = 1+3$

$\mathbb{N} = \{1, 2, 3, \ldots\ldots\}$

$0 < 1 < 2 < 3 < \ldots\ldots$

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\ldots\}$

$\cdots < -2 < -1 < 0 < 1 < 2 < \cdots\cdots$

$\mathbb{Q} = \left\{ \frac{a}{b} : a \text{ and } b \in \mathbb{Z} \text{ and } b \neq 0 \right\}$

<u>Def</u>: $n \in \mathbb{N}$ $a \in \mathbb{R}$ then $a^n = \underbrace{a \cdot a \cdot \cdots\cdots \cdot a}_{n \text{ times}}$

If $a \neq 0$, $a^0 = 1$

If $a \neq 0$, $a^{-n} = \frac{1}{a^n} = (a^n)^{-1}$

<u>Properties</u>: 1) $a^n a^m = a^{n+m}$

2) $(a^n)^m = a^{nm}$

3) $(ab)^n = a^n b^n$

Def: $a \in \mathbb{R}$. The absolute value of a is

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Properties: 1) $|a| \geq 0$ and $|a| = 0 \iff a = 0$

2) $|ab| = |a||b|$ $\quad \forall\, a, b \in \mathbb{R}$

3) $|a|^2 = a^2$

4) $|a+b| \leq |a| + |b|$

proof. $\left.\begin{array}{l} -|a| \leq a \leq |a| \\ -|b| \leq b \leq |b| \end{array}\right\} \Rightarrow \begin{array}{l} -|a|-|b| \leq a+b \leq |a|+|b| \\ -(|a|+|b|) \leq a+b \leq |a|+|b| \quad \Rightarrow \end{array}$

$\Rightarrow \quad |a+b| \leq |a|+|b|$

We have used $\quad -c \leq x \leq c \Rightarrow |x| \leq c$

proof If $x \geq 0 \quad |x| = x \leq c$ ✓

If $x < 0 \Rightarrow |x| = -x \qquad$ since $-c \leq x \Rightarrow x+c \geq 0$

$\Rightarrow -x-c \leq 0 \Rightarrow \underbrace{-x}_{\substack{\| \\ |x|}} \leq c \Rightarrow |x| \leq c$ ✓

5) $\big||a|-|b|\big| \leq |a-b|$

proof

$$\left.\begin{array}{l} |x+y| \le |x| + |y| \\ x = a-b \quad y = b \end{array}\right\} \Longrightarrow \quad |a| \le |a-b| + |b|$$

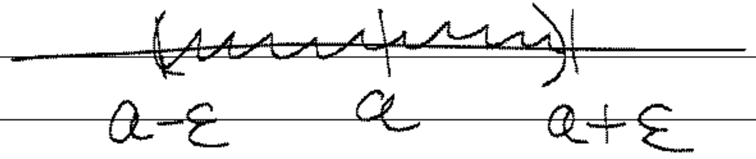
$$|a| - |b| \le |a-b|$$

$$x = b-a \quad y = a \quad \Longrightarrow \quad |b| - |a| \le |b-a| = |a-b|$$

$$-|a-b| \le |a| - |b| \le |a-b| \Rightarrow \big||a|-|b|\big| \le |a-b|$$

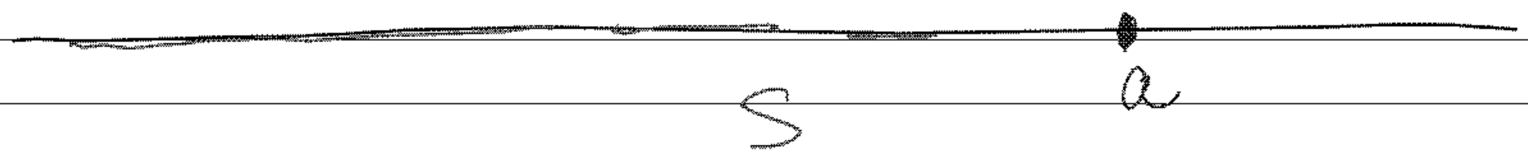
Obs: $|a_1 + a_2 + \cdots + a_n| \le |a_1| + |a_2| + \cdots + |a_n|$

Obs. $|x-a| < \varepsilon \iff -\varepsilon < x-a < \varepsilon \iff a-\varepsilon < x < a+\varepsilon$



$\boxed{\text{Section 2.3}}$

Def: 1) $S \subset \mathbb{R}$. An upper bound for the set S is a number a such that $s \le a \quad \forall\, s \in S$.

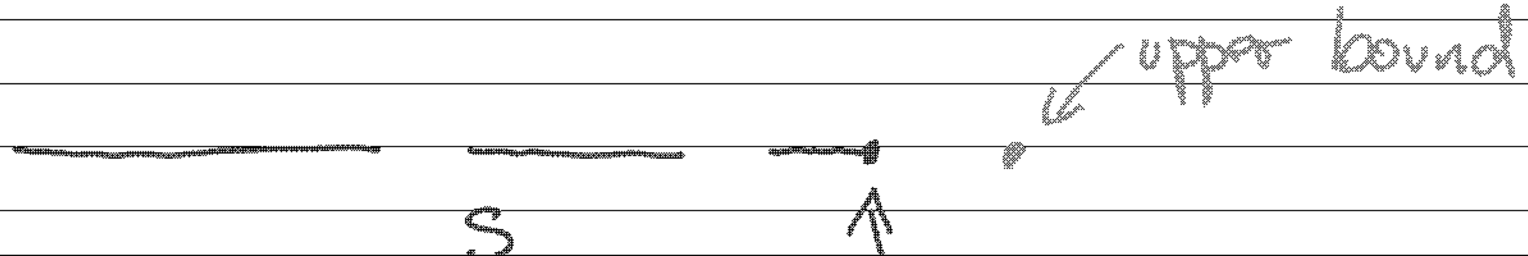


2) If S has an upperbound, we say S is bounded from above

3) y is a least upper bound of S if

a) y is an upper bound of S

b) if b is an upper bound of S, then $y \le b$



least upper bound

Notation: lub

Obs. least upper bounds are unique

proof y_1 and y_2 are two least upper bounds of S then

$$\left.\begin{array}{l} y_1 \text{ lub } \& \ y_2 \text{ ub} \Rightarrow y_1 \le y_2 \\ y_1 \text{ ub } \& \ y_2 \text{ lub} \Rightarrow y_2 \le y_1 \end{array}\right\} \Rightarrow y_1 = y_2$$

Obs: y lub of S. $x < y \Rightarrow \exists\, s \in S \quad x < s \le y$

proof: y lub and $x < y \Rightarrow x$ is not a ub $\Rightarrow \exists\, s \in S: x < s$

but, since y ub we also have $s \le y \Rightarrow x < s \le y$