Note Title 1/27/2016

<u>Def</u>: $S \subset \mathbb{R}$

max$S$ is a number that satisfies

1) it is an upper bound of S

2) it belongs to S

<u>Obs</u>: max$S$ need not exists

<u>Obs</u>: If max$S$ exists, then max$S$ is lub of S. In particular there is at most one max$S$

<u>Ex</u>: 1) $\mathbb{Z}$ no upper or lower bound

2) $(0,1) = \{x : 0 < x < 1\}$

no max, no min (defined similarly as max)

lub $(0,1) = 1$

Obs: If S is finite $\Rightarrow$ $\max S$ exists

Property: If $S \neq \emptyset$ and S is bounded from above, the S has a lub.

Example: 1) $S = \{x \in \mathbb{R} : x < \sqrt{2}\}$

lub $= \sqrt{2}$

2) $S = \{x \in \mathbb{Q} : x < \sqrt{2}\}$ $\quad S \subset \mathbb{Q}$

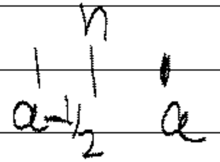
$S \subset \mathbb{R}$ $\quad$ lub $S = \sqrt{2}$

Notation: glb S = greatest lower bound of S

Obs: $\forall x \in \mathbb{R}$ $\exists n \in \mathbb{N}$ such that $n > x$.

proof: Assume $n \le x$ $\forall n \in \mathbb{N}$. then x is an ub. thus $\mathbb{N}$ is bounded from above. thus $\exists$ $a = \text{lub } \mathbb{N}$. thus $\exists$ $n \in \mathbb{N}$ such that

$$a - \frac{1}{2} < n \le a \implies a < a + \frac{1}{2} < n + 1 \in \mathbb{N}$$



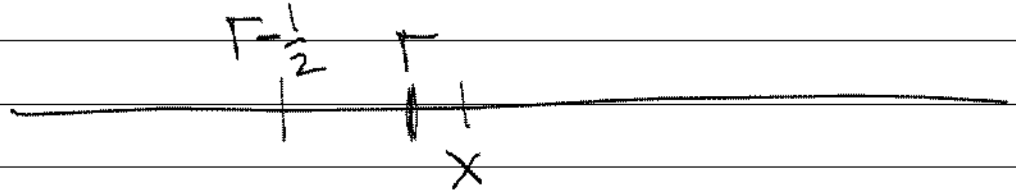
contradiction

Obs: $\forall \varepsilon > 0$ $\exists$ $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$

proof: $\exists$ n such that $n \in \mathbb{N}$ and $n > \frac{1}{\varepsilon} \implies \frac{1}{n} < \varepsilon$

Obs: $\forall x\in\mathbb{R}$ $\exists n\in\mathbb{Z}$ such that $n\le x<n+1$

proof



$S=\{n\in\mathbb{Z};\ n\le x\}$ Let $r=\text{lub}\,S$

$\exists!\,\bar{n}\in[r-\frac{1}{2}, r]$ $\quad n\le\bar{n}\le r$ $\quad\forall n\in S \Rightarrow \bar{n}=\text{lub}\,S \Rightarrow r=\bar{n}$

$n=r=\bar{n}$ $\quad\boxed{n\le x}$ $\quad n=\text{lub}\,S$ $\quad n+1>n\Rightarrow n+1\notin S\Rightarrow \boxed{n+1>x}$

Obs: $\forall x\in\mathbb{R}$ and $\forall m\in\mathbb{N}$ $\exists n\in\mathbb{Z}$ such that

$$\frac{n}{m}\le x<\frac{n+1}{m}$$

proof: Apply prev obs to xm to conclude $\exists n \in \mathbb{Z}$ such that $n \leq xm < n+1$. Devide by m to get

$$\frac{n}{m} \leq x < \frac{n+1}{m}$$

Obs: $\forall x \in \mathbb{R}$ $\forall \varepsilon > 0$ $\exists r \in \mathbb{Q}$ such that $|x-r| < \varepsilon$

proof: Use prev obss $\exists m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$ $\exists n \in \mathbb{Z}$ such that $\frac{n}{m} \leq x < \frac{n+1}{m}$ substract $\frac{n}{m}$ set $r = \frac{n}{m}$

$$0 \leq x - r = |x-r| < \frac{1}{m} < \varepsilon \quad \checkmark$$

Notation $a_1, \ldots\ldots, a_n \in \{0, 1, \ldots, 9\}$ $a_0 \in \mathbb{Z}$

$$a_0.a_1a_2\ldots\ldots a_n = a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n}$$

Ex: $3.75 = 3 + \frac{7}{10} + \frac{5}{10^2}$

$$(-2).75 = -2 + \frac{7}{10} + \frac{5}{10^2}$$