Note Title 1/29/2016

$a_0 \in \mathbb{Z} \qquad a_i \in \{0, 1, \dots, 9\}$

$$a_0 . a_1 a_2 \dots a_n = a_0 + \sum_{i=1}^{n} \frac{a_i}{10^i}$$

Obs: $m < n \qquad a_0 . a_1 a_2 \dots a_m \leq a_0 . a_1 \dots a_m \dots a_n < a_0 . a_1 \dots a_m + 10^{-m}$

proof: $\frac{a_{m+1}}{10^{m+1}} + \qquad + \frac{a_n}{10^n} \leq \frac{9}{10^{m+1}} + \dots + \frac{9}{10^n} = \frac{9}{10^{m+1}}\left(1 + \frac{1}{10} + \dots + \frac{1}{10^{n-m-1}}\right) <$

$$< \frac{9}{10^{m+1}} \qquad \frac{1}{1 - \frac{1}{10}} = \frac{9}{10^{m+1}} \quad \frac{10}{9} = \frac{1}{10^m} = 10^{-m}$$

Assume $a_0 \in \mathbb{Z} \quad a_i \in \{0, \dots, 9\} \quad i \in \mathbb{N}$ given and fixed

$S = \{ a_0.a_1 \ldots a_n : n \in \mathbb{N} \}$

Example: $a_0 = 3 \qquad a_1 = a_3 = a_5 = \ldots = 1 \qquad a_2 = a_4 = a_6 = \ldots = 2$

$S = \{ 3.1, 3.12, 3.121, 3.1212, \ldots \}$

Obs: For any n, $a_0.a_1 \ldots a_n + 10^{-n}$ is an upper bound of S.

Def: Let $a = \text{lub } S$. Then $a_0.a_1 \ldots a_n \ldots$ is called a decimal expansion of a.

Obs: $a_0.a_1 \ldots a_n \leq a \leq a_0.a_1 \ldots a_n + 10^{-n}$ for all n

Obs: $5.13999\ldots = 5.14000\ldots$

Prop: Let $a > 0 \Rightarrow \exists!\, b > 0$ such that $b^2 = a$

proof: $0 < x_1 < x_2 \Rightarrow x_1^2 < x_2^2$. This proves uniqueness

$S = \{ x \in \mathbb{R} : x > 0 \text{ and } x^2 \leq a \}$

$$\left.\begin{array}{l} \min\{1, a\} \in S \quad \text{Thus } S \neq \emptyset \\ a+1 \text{ is an upper bound of } S. \end{array}\right\} \Rightarrow S \text{ has a lub}$$

Let $b = \text{lub } S$

Let $\varepsilon > 0$ and $\varepsilon < b \Rightarrow (b-\varepsilon)^2 < b^2 < (b+\varepsilon)^2$

$b+\varepsilon \notin S$ because b is an upper bound of S and $b+\varepsilon > b$

thus $(b+\varepsilon)^2 > a$

$(b-\varepsilon)^2 < a$ because if $(b-\varepsilon)^2 \geq a$ then $b-\varepsilon$ would be an upper bound of S.

Then $\quad (b-\varepsilon)^2 < a < (b+\varepsilon)^2$

$$+\quad -(b+\varepsilon)^2 < -b^2 < -(b-\varepsilon)^2$$

$$(b-\varepsilon)^2-(b+\varepsilon)^2 < a-b^2 < (b+\varepsilon)^2-(b-\varepsilon)^2$$

$$-4b\varepsilon < a-b^2 < 4b\varepsilon$$

$$|a-b^2| < 4b\varepsilon \qquad \forall \varepsilon > 0$$

If $|a-b^2| \neq 0$, take $\varepsilon = \frac{|a-b^2|}{4b}$ to get $|a-b^2| < \cancel{4b}\,\frac{|a-b|^2}{\cancel{4b}}$ impossible

Thus $a=b^2$

<u>Notation</u>: $b=\sqrt{a}$

Metric spaces

Def: A metric space is a set E together with $d: E\times E \to \mathbb{R}$ such that

1) $d(p,q) \geq 0 \quad \forall\, p,q \in E$ and $d(p,q) = 0 \iff p = q$
2) $d(p,q) = d(q,p) \quad \forall\, p,q \in E$
3) $d(p,r) \leq d(p,q) + d(q,r) \quad \forall\, p,q,r \in E$

Notation: (E,d) $\quad d(p,q)$ = distance from p to q

Example: 1) $E = \mathbb{R}$ $\quad d(p,q) = |p-q|$

proof: 1) ✓ $\quad$ 2) ✓ $\quad$ 3) ✓