Note Title 2/1/2016

Metric spaces

Example: $\mathbb{R}^n = \{(a_1, \dots, a_n) : a_i \in \mathbb{R} \quad 1 \le i \le n\}$

$x = (x_1, \dots, x_n) \qquad y = (y_1, \dots, y_n)$

$d(x,y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$

Proposition (Schwartz inequality)

$a_i, b_i \quad 1 \le i \le n$ then

$$\left| \sum_{i=1}^{n} a_i b_i \right| \le \sqrt{\sum_{i=1}^{n} a_i^2} \sqrt{\sum_{i=1}^{n} b_i^2}$$

Proof: Let $\alpha, \beta \in \mathbb{R}$

$$0 \leq \sum_{i=1}^{n} (\alpha a_i - \beta b_i)^2 = \alpha^2 \sum_{i=1}^{n} a_i^2 - 2\alpha\beta \sum_{i=1}^{n} a_i b_i + \beta^2 \sum_{i=1}^{n} b_i^2$$

Set $\alpha = \sqrt{\sum_{i=1}^{n} b_i^2}$ $\quad \beta = \pm \sqrt{\sum_{i=1}^{n} a_i^2}$

$$0 \leq 2\left(\sum_{i=1}^{n} b_i^2\right)\left(\sum_{i=1}^{n} a_i^2\right) \mp 2 \sqrt{\sum_{i=1}^{n} b_i^2} \sqrt{\sum_{i=1}^{n} a_i^2} \left(\sum_{i=1}^{n} a_i b_i\right)$$

$$\pm \left(\sum_{i=1}^{n} a_i b_i\right) \leq \sqrt{\sum_{i=1}^{n} a_i^2} \sqrt{\sum_{i=1}^{n} b_i^2} \quad \checkmark$$

<u>Corollary</u> $\sqrt{\sum_{i=1}^{n} (a_i + b_i)^2} \leq \sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2}$

~~proof~~: $\sum_{i=1}^{n}(a_i+b_i)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{n} b_i^2 \leq$

$\leq \sum_{i=1}^{n} a_i^2 + 2\sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2} + \sum_{i=1}^{n} b_i^2 = \left(\sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2}\right)^2$

Back to show that $d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}$ is a distance.

Let $x, y, z \in \mathbb{R}^n$

Set $a_i = x_i - z_i$ $\quad b_i = z_i - y_j$ appy corollary

$d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2} \leq \sqrt{\sum_{i=1}^{n}(x_i-z_i)^2} + \sqrt{\sum_{i=1}^{n}(z_i-y_i)^2} = d(x,z) + d(z,y)$ ✓

this shows that $d$ is a distance

More examples 1) $(E,d)$ a metric space, $E' \subset E$, then $(E',d)$ is also a metric space

2) $E$ any set $\quad d(x,y)=\begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } x\neq y \end{cases}$

$x, y, z$

if $x \neq y$ $\quad \underset{\underset{1}{\parallel}}{d(x,y)} \leq \underbrace{d(x,z)+d(z,y)}_{=1 \text{ or } 2}$ ✓

Prop: $x_1, \ldots, x_n \in E$. $(E,d)$ a metric space

$$d(x_1,x_n) \leq d(x_1,x_2)+d(x_2,x_3)+\cdots+d(x_{n-1},x_n)$$

proof: Induction

$n=2 \qquad d(x_1,x_2) \le d(x_1,x_2)$ ✓

$n>2 \qquad d(x_1,x_n) \le d(x_1,x_{n-1}) + d(x_{n-1},x_n) \le d(x_1,x_2) + d(x_2,x_3) + \dots$

bec $\Delta$ ineq $\qquad\qquad$ by induction

$\dots + d(x_{n-2},x_{n-1}) + d(x_{n-1},x_n)$ ✓

<u>Prop</u>: $x,y,z \in E$, $(E,d)$ metric then

$$|d(x,z)-d(y,z)| \le d(x,y)$$

<u>proof</u>: $d(x,z) \le d(x,y) + d(y,z) \Rightarrow d(x,z) - d(y,z) \le d(x,y)$

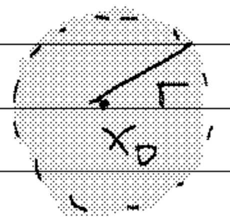
$d(y,z) \le d(y,x) + d(x,z) \Rightarrow d(y,z) - d(x,z) \le d(x,y)$ $\Big\} =$

$$\Rightarrow \quad |d(x,z)-d(y,z)| \le d(x,y)$$

## Open and closed sets

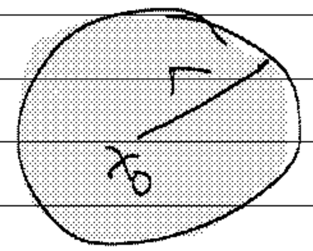
**Def:** $(E, d)$ metric space. $x_0 \in E$, $r > 0$. The open ball of radius $r$ centered at $x_0$ is

$$B_r(x_0) = \{ x \in E : d(x, x_0) < r \}$$




The closed ball of radius $r$ centered at $x_0$ is

$$C_r(x_0) = \{ x \in E : d(x, x_0) \le r \}$$




**Def:** $a, b \in \mathbb{R}$ $\quad a < b$

$(a,b) = \{x \in \mathbb{R} : a < x < b\}$ is called an open interval

$[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ is called closed interval.