Note Title 2/5/2016

Prop: 1) E is closed

2) $\emptyset$ is closed

3) the intersection of closed sets is closed

4) the union of a finite number of closed sets is closed

proof: 3) C_α $\alpha \in I$ closed each of them

$$\left(\bigcap_{\alpha \in I} C_\alpha\right)^c = \underbrace{\bigcup_{\alpha \in I} \underset{\substack{\uparrow \\ \text{open}}}{C_\alpha^c}}_{\text{open}} \Longrightarrow \bigcap_{\alpha \in I} C_\alpha \text{ is closed}$$

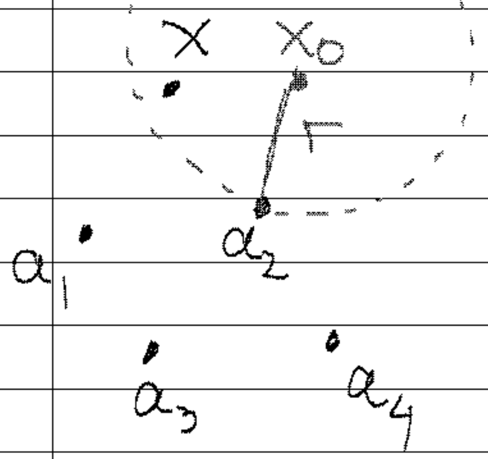
4) C_i $i=1,\dots,n$ closed $\left(\bigcup_{i=1}^{n} C_i\right)^c = \underbrace{\bigcap_{i=1}^{n} \underbrace{C_i^c}_{\text{open}}}_{\text{open}} \Longrightarrow$

$\bigcup_{i=1}^{n} C_i$ is closed.

<u>Ex</u>: $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$

$\bigcup_{n=1}^{\infty} \left[\left(-\infty, -\frac{1}{n}\right] \cup \left[\frac{1}{n}, \infty\right)\right] = (-\infty, 0) \cup (0, +\infty)$ open and not closed.

Obs: S finite $\Rightarrow$ S is closed



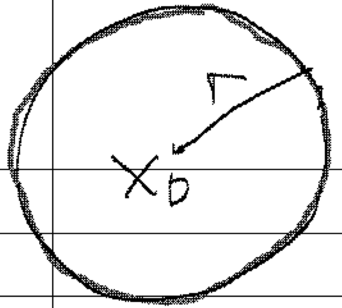
proof: Need to show S^c is open

Let $x_0 \in S^c$, $S = \{a_1, a_2, \dots, a_n\}$

Let $r = \min_{1 \le i \le n} d(x_0, a_i)$

If $d(x, x_0) < r \Rightarrow x \neq a_i$ for $1 \le i \le n$ since otherwise, if $x = a_\ell$ then $d(x, x_0) = d(a_\ell, x_0) \ge r$. Contradiction. Thus $B_r(x_0) \subset S^c \Rightarrow S^c$ is open $\Rightarrow S$ closed.

Obs: $x_0 \in E$, $r > 0$ $\{x \in E : d(x, x_0) = r\}$ is closed



proof: $\underbrace{\{x\in E : d(x,x_0)\le r\}}_{\text{closed}} \cap \underbrace{\{x\in E : d(x,x_0)< r\}^c}_{\text{closed}}$

$\{x\in E : d(x,x_0)< r\}^c = \{x\in E : d(x,x_0)\ge r\}$

$= \underbrace{\{x\in E : d(x,x_0)= r\}}_{\text{is closed}}$

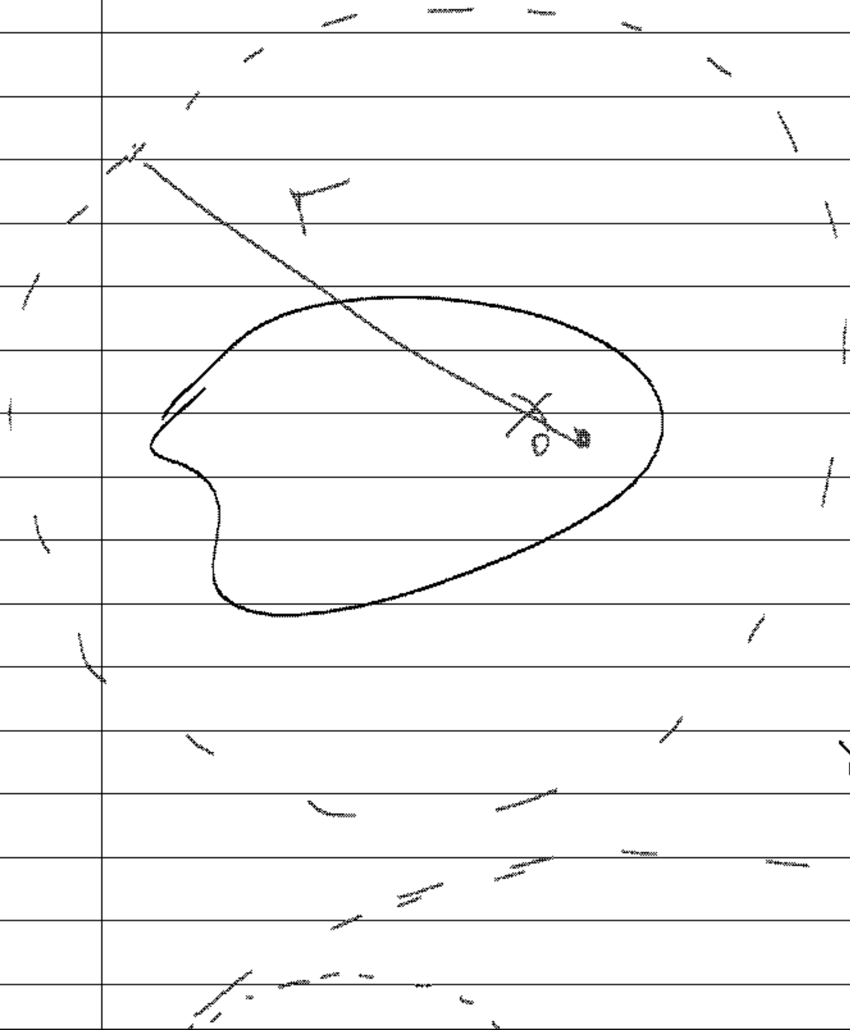
Obs. $[a,b)$ $a,b\in\mathbb{R}$

neither closed nor open

Names: $(a_1,b_1)\times\ldots\ldots\times(a_n,b_n)$ open interval in $\mathbb{R}^n$

$[a_1,b_1]\times\ldots\ldots\times[a_n,b_n]$ closed interval in $\mathbb{R}^n$

$(a_1,b_1)\times(a_2,b_2) = \{(x,y) : a_1 < x < b_1 \text{ and } a_2 < y < b_2\}$



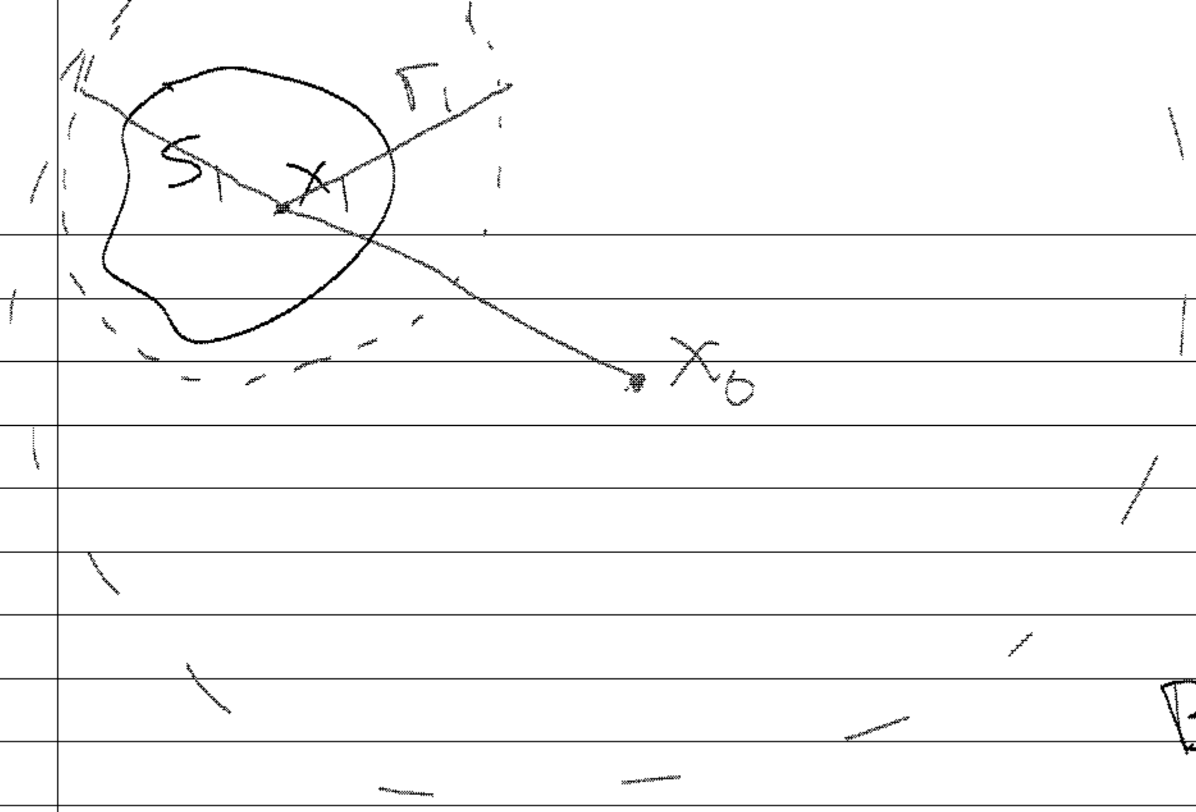
Def: S is bounded if $\exists\, x_0 \in E$ and $r>0$ such that $S \subset B_r(x_0)$

Obs: $S_1, S_2, \dots, S_n$ bounded then $\bigcup_{i=1}^{n} S_i = S$ is bounded

proof: S_i bounded $\Rightarrow S_i \subset B_{r_i}(x_i)$

Let $x_0 \in E$, Note

$$B_{r_i + d(x_i,x_0)}(x_0) \supset B_{r_i}(x_i) \supset S_i$$



Let $r = \max\limits_{1 \le i \le n} r_i + d(x_i, x_0)$

$B_r(x_0) \supset B_{r_i + d(x_i,x_0)}(x_0) \supset S_i \quad \forall i$

$B_r(x_0) \supset S = \bigcup\limits_{i=1}^{n} S_i$

<u>Prop</u>: $S \neq \emptyset \quad S \subset \mathbb{R} \quad S$ closed

1) If S is bounded from above, $\exists\, x \in S$ such that $y \le x \quad \forall y \in S$

<u>proof</u>: Let $x = \text{lub } S$

If $x \in S^c$, since S^c is open, $\exists\ \varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset S^c \Rightarrow$ $x-\varepsilon$ is also an upper bound of S

Contraction $\Rightarrow x \in S$.

2) If S is bounded from below $\exists\ x \in S$: $y \geq x \quad \forall y \in S$