$S_k \neq \phi$

Note Title 2/22/2016

$S_1 \supset S_2 \supset S_3 \supset \cdots\cdots$

$\bigcap_{k=1}^{\infty} S_k = \phi$ It could be either

or $\neq \phi$

$S_k = (k, \infty)$ $S_k = [0, 1]$

<u>Prop</u>: E compact $E \supset S_1 \supset S_2 \supset \cdots \supset S_n \supset \cdots$ $S_n \neq \phi\ \forall n$

S_n is closed $\forall n$. then $\bigcap_{n=1}^{\infty} S_n \neq \phi$

<u>proof</u>: If $\bigcap_{n=1}^{\infty} S_n = \phi$. then $\left(\bigcap_{n=1}^{\infty} S_n\right)^c = \phi^c = E$

$$\bigcup_{n=1}^{\infty} S_n^c = E \underset{\substack{E \text{ compact} \\ S_n \text{ are closed}}}{\Longrightarrow} \exists N: \bigcup_{n=1}^{N} S_n^c = E \Rightarrow \bigcap_{n=1}^{N} S_n = \phi \Rightarrow$$

$$\Rightarrow S_N = \phi \text{ contradiction}$$

<u>Def</u>: $S \subset E$. $x \in E$ is a cluster point of S if $\forall \varepsilon > 0$ $B_\varepsilon(x) \cap S$ contains an infinite number of points.

<u>th</u>: $S \subset E$, E compact, $\#(S) = \infty$. then S has a cluster point.

<u>proof</u>: Assume S has no cluster point. $\forall x \in E$ $\exists \varepsilon_x > 0$

such that $\#(S \cap B_{\varepsilon_x}(x)) < \infty$. $E = \bigcup_{x \in E} B_{\varepsilon_x}(x)$. Since

E is compact and $B_{\varepsilon_x}(x)$ are open $\exists$ $x_1, \ldots, x_r$ such that

$$E = \bigcup_{i=1}^{r} B_{\varepsilon_i}(x_i) \qquad S = S \cap E = S \cap \left(\bigcup_{i=1}^{r} B_{\varepsilon_i}(x_i) \right) =$$

$$= \bigcup_{i=1}^{r} \left(S \cap B_{\varepsilon_i}(x_i) \right)$$

$$\#(S \cap B_{\varepsilon_i}(x_i)) = n_i \quad \Rightarrow \#(S) \leq \sum_{i=1}^{r} \#(S \cap B_{\varepsilon_i}(x_i)) = n_1 + \cdots + n_r < \infty$$

Contradiction

Corollary: x_n a sequence in E. E compact $\Rightarrow \exists$ a subsequence that converges

pf: $S = \{x_1, x_2, \ldots, x_n, \ldots\}$

if $\#(S) < \infty$. $\exists\, a \in E$ and $1 \le n_1 < n_2 < \ldots < n_r < \ldots$ such that

$x_{n_1} = x_{n_2} = \ldots = x_{n_r} = \ldots = a \Rightarrow \quad x_{n_r} \to a$

if $\#S = \infty \Rightarrow \exists\, a \in E$ cluster point of $S \Rightarrow B_{\frac{1}{n}}(a) \cap S$ has an infinite number of elements.

Let n_1: $x_{n_1} \in B_{1/1}(a) \cap S$

$n_2 > n_1$: $x_{n_2} \in B_{1/2}(a) \cap S$

$\vdots$

$n_{r+1} > n_r$: $x_{n_{r+1}} \in B_{\frac{1}{r+1}}(a) \cap S$

$x_{n_r} \to a$ (you can prove it)

<u>Corollary</u>: E compact $\Rightarrow$ E complete

<u>proof</u>: x_n Cauchy. Since E is compact, $\exists\, a$ and x_{n_r} such that $x_{n_r} \to a$. We know that a Cauchy sequence with a convergent subsequence, converges. $\Rightarrow$ $x_n \to a$.

$\Rightarrow$ E is complete.

Corollary: $S \subset E$. S compact $\Rightarrow$ S is closed

proof: S compact $\Rightarrow$ S complete $\Rightarrow$ S is closed

Obs: S complete $\Rightarrow$ S is closed (you can do it)