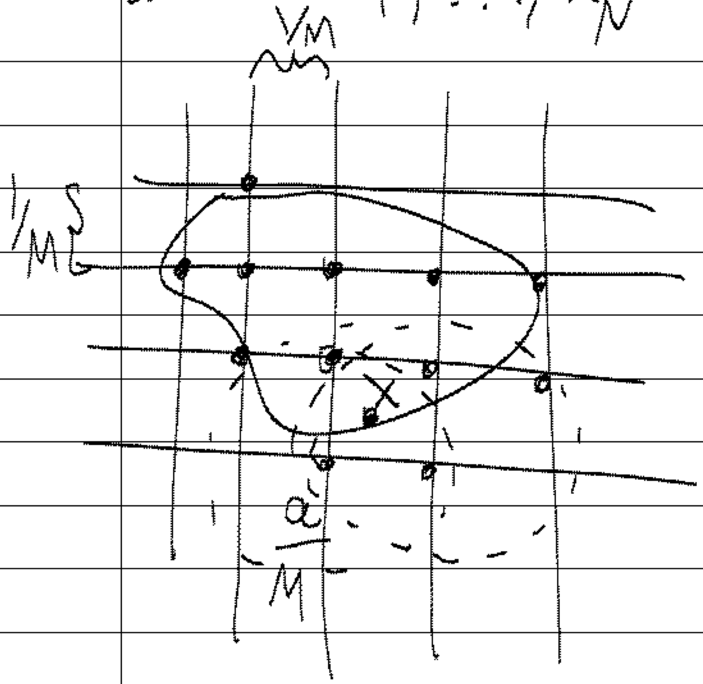
Note Title 2/24/2016

<u>Lemma</u> $S \subseteq \mathbb{R}^n$, S bounded $\Rightarrow \forall \varepsilon > 0 \; \exists N$ and $x_1, \dots, x_N$ such that $S \subset \bigcup_{i=1}^{N} \{ d(x, x_i) \le \varepsilon \}$

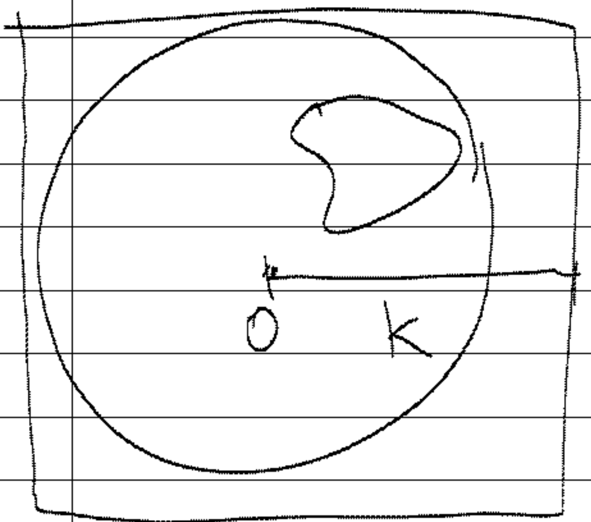


<u>proof</u>: M positive integer, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. $\exists\, a = (a_1, \dots, a_n) \in \mathbb{Z}^n$ such that

$$\frac{a_i}{M} \le x_i < \frac{a_i + 1}{M} \Rightarrow d\left(x, \frac{a}{M}\right) = \sqrt{\sum_{i=1}^{n} \left(x_i - \frac{a_i}{M}\right)^2}$$

$$\le \sqrt{\sum_{i=1}^{n} \left(\frac{1}{M}\right)^2} = \frac{\sqrt{n}}{M}. \quad \text{Pick } M \text{ such that } \frac{\sqrt{n}}{M} < \varepsilon$$

S bounded $\Rightarrow \exists\, K > 0$ such that $|x_i| \le K \quad \forall x \in S$.



$$[-K, K]^n \supset S$$

$$[-K, K]^n \subset \bigcup_{a \in \mathbb{Z}^n} B_\varepsilon\left(\frac{a}{M}\right)$$

$$|a_i| \le KM + \sqrt{n}$$

$$|x_i| \le K$$

$$\left|\frac{a_i}{M}\right| \le K + \frac{\sqrt{n}}{M}$$

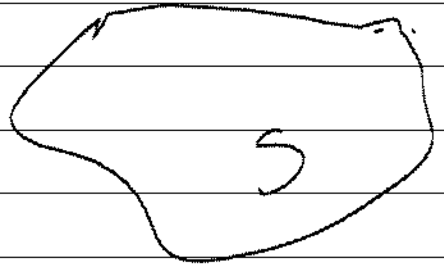
<u>Theorem</u>: $S \subset \mathbb{R}^n$. S closed & bounded $\Rightarrow$ S compact

<u>proof</u>. Let $S \subset \bigcup_{i \in I} U_i$ $\quad U_i$ open

We want to show that $\exists$ r positive integer and $i_1, i_2, \dots, i_r$ such that

$$S \subset \bigcup_{j=1}^{r} U_{i_j}$$

Assume this is not true. $\varepsilon_1 = \frac{1}{2}$ apply prev lemma to



get $S = \bigcup_{i=1}^{N} \left(C_{\frac{1}{2}}(x_i) \cap S \right) \subset \bigcup_{i \in I} U_i$

$$\boxed{C_\varepsilon(x_0) = \{x : d(x, x_0) \le \varepsilon\}}$$

$\exists\, 1 \le i \le N$ such that $S_2 = C_{1/2}(x_i) \cap S$ can not be covered by a finite number of the U_is. Note S_2 is closed.

Proceed inductively to get

$S \supset S_2 \supset S_3 \supset S_4 \supset \dots$

$S_j \subset C_{1/j}(z_j)$ for some z_j

S_j can not be covered by a finite number of the U_is.

Let $x_j \in S_j$. Let $\varepsilon > 0$. If $n, k \geq N$ $d(x_n, x_k)$

note $x_n \in S_n \subset S_N \subset C_{1/N}(z_N)$

$x_k \in S_k \subset S_N \subset C_{1/N}(z_N)$

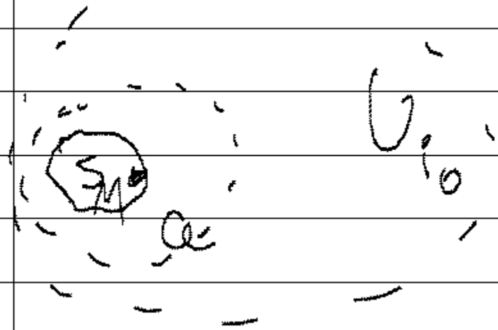
$d(x_n, x_k) \leq d(x_n, z_N) + d(x_k, z_N) \leq \frac{1}{N} + \frac{1}{N} = \frac{2}{N} < \varepsilon$

as long as $N > \frac{2}{\varepsilon}$.

Thus x_n is Cauchy. Since $\mathbb{R}^n$ is complete $x_n \longrightarrow a$

$x_n \in S_n \subset S$ closed $\Rightarrow$ $a \in S \subset \bigcup_{i \in I} U_i \Rightarrow \exists\, i_0 \in I$

such that $a \in V_{i_0} \Rightarrow \exists\, \varepsilon > 0$ such that $B_\varepsilon(a) \subset V_{i_0}$



note. Given any M, if $n \geq M$

$x_n \in S_n \subset S_M \quad \forall n \Rightarrow a \in S_M \quad \forall M$

$a \in S_M \subset C_{\frac{1}{M}}(z_M) \Rightarrow$ if $x \in S_M$ $\quad d(x,a) \leq \frac{2}{M}$ so

if $M > \frac{2}{\varepsilon} \Rightarrow S_M \subset B_\varepsilon(a) \subset V_{i_0}$. Contradiction