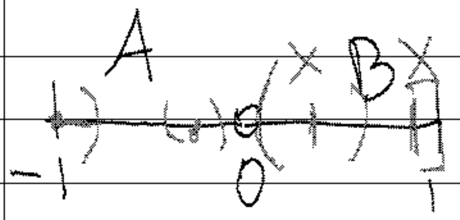
Note Title 2/26/2016

Connectedness

Def: E is connected if E and ϕ are only sets that are both open and closed.

Obs: E is not connected $\iff$ $\exists$ A, B open sets such that $A \cap B = \phi$, $A \neq \phi$ and $B \neq \phi$ and $A \cup B = E$

Ex: $E = [-1,0) \cup (0,1]$ is not conneted.



$A = [-1,0)$ is both open and closed in E

$B = (0,1]$ is open $B = A^c \Rightarrow A$ is closed

Let $x \in B$, $x > 0$. If $d(y,x) < x \Rightarrow |y-x| < x \Rightarrow$
$-x < y-x < x \Rightarrow 0 < y \Rightarrow y \in B \Rightarrow B_x(x) \subset B$

Ex: 1) $\mathbb{N}$ is not connected

2) $\mathbb{Q}$ is not connected $\mathbb{Q} = \{x \in \mathbb{Q} : x < \sqrt{2}\} \cup \{x \in \mathbb{Q} : x > \sqrt{2}\}$

Obs: E metric. $S \subset E$. Then S is also metric, is called a subspace. 1) If $U \subset E$ and U open in $E \Rightarrow U \cap S$ is open in S. 2) If $U \subset S$ and U is open in $S \Rightarrow \exists V \subset E$: V is open in E and $U = V \cap S$

<u>Example</u>: 1) $E = \mathbb{R}$ $\quad S = [0,1) \cup (1,2]$

$U = [0,1) \subset S$ $\quad$ U is open in S

$U = \underbrace{(-1,1)}_{=V} \cap S$ with V open in $\mathbb{R}$

2) $\underset{\text{open in } \mathbb{Q}}{\{x \in \mathbb{Q} : x < \sqrt{2}\}} = \underset{\text{open in } \mathbb{R}}{\{x \in \mathbb{R} : x < \sqrt{2}\}} \cap \mathbb{Q}$

<u>Th</u>: E metric space. $S \subset E$. S is not connected $\Longleftrightarrow$ $\exists A, B$ both open in E such that $A \cap B = \phi$, $A \cap S \neq \phi$,

$B \cap S \neq \emptyset$ and $S \subset A \dot{\cup} B$

Example: 1) $S = [-1,0) \cup (0,1] \subset \mathbb{R}$

$A = (-2,0)$ $B = (0,2)$ $A \cap B = \emptyset$

$S \subset A \dot{\cup} B$ $A \cap S \neq \emptyset$ $B \cap S \neq \emptyset$

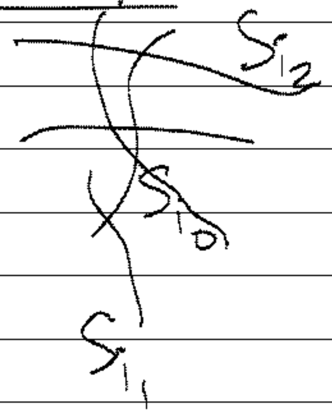
Prop: $S \subset \mathbb{R}$. Assume $\exists\, a, b \in S$ and $a < c < b$ such that $c \notin S \Rightarrow S$ is not connected.

proof: $S = \underbrace{(S \cap \{x < c\})}_{\neq \emptyset} \dot{\cup} \underbrace{(S \cap \{x > c\})}_{\neq \emptyset}$

Prop: $\{S_i\}_{i\in I}$ $S_i \subset E$. S_i is connected $\forall i \in I$

$\exists S_{i_0} : S_i \cap S_{i_0} \neq \emptyset \quad \forall i \in I. \Rightarrow S = \bigcup_{i\in I} S_i$ is connected

proof



Assume $S = A \,\dot\cup\, B$ A, B open in S

S_{i_0} is connected $\Rightarrow S_{i_0} \subset A$ or B.

Assume $S_{i_0} \subset A$

Let $i \in I \Rightarrow S_i \subset A$ or B.

but $\emptyset \neq S_{i_0} \cap S_i \subset A$ then $S_{i_0} \cap S_i \not\subset B \Rightarrow S_i \subset A$

This is $\forall i \in I \Rightarrow S \subset A$.