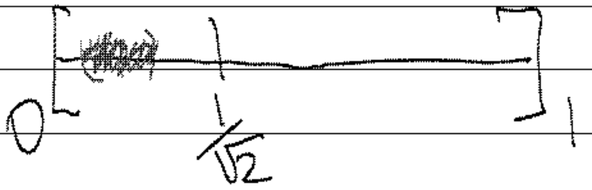
<u>Theorem</u>: $S \subset E$. S is not connected $\iff \exists\, A, B$ open in E such that $A \cup B \supset S$, $A \cap S \neq \emptyset$, $B \cap S \neq \emptyset$, $A \cap B = \emptyset$.

<u>proof</u>: $\Rightarrow$) S not connected $\Rightarrow \exists\, \emptyset \subsetneq A' \subsetneq S$ such that A' and $B' = S - A'$ such that both A' and B' are open

<u>Ex</u>: open $A' = [0, \frac{1}{\sqrt{2}}) \cap \mathbb{Q}$ $\quad B' = (\frac{1}{\sqrt{2}}, 1] \cap \mathbb{Q}$ both A' and B' are open in $[0,1] \cap \mathbb{Q}$ $\quad B' = ([0,1] \cap \mathbb{Q}) - A'$



Let $x \in A'$,

since A' is open in S $\exists\, \varepsilon_x > 0$ such that $B_{\varepsilon_x}(x) \cap S \subset A'$

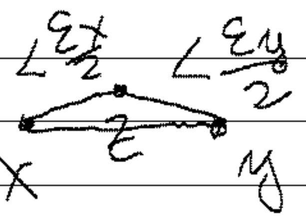
Similarly $\forall\, y \in B'$, $\exists\, \varepsilon_y > 0$ such that $B_{\varepsilon_y}(y) \cap S \subset B'$

Let $A = \bigcup_{x \in A'} B_{\frac{\varepsilon_x}{2}}(x)$ $\qquad B_{\frac{\varepsilon_x}{2}}(x) = \{ z \in E : d(z,x) < \frac{\varepsilon_x}{2} \}$

$B = \bigcup_{y \in B'} B_{\frac{\varepsilon_y}{2}}(y)$

$A \cap S = A'$ and $B \cap S = B'$, A and B are open in E

Let $z \in A \cap B$. Then $\exists\, x \in A'$ such that $z \in B_{\frac{\varepsilon_x}{2}}(x)$
and $\exists\, y \in B'$ such that $z \in B_{\frac{\varepsilon_y}{2}}(y)$



Then $d(x,y) \le d(x,z) + d(z,y) < \frac{\varepsilon_x}{2} + \frac{\varepsilon_y}{2} \le \varepsilon_x$

Assume $\varepsilon_x \geq \varepsilon_y$

$d(x,y) < \varepsilon_x \Rightarrow y \in B_{\varepsilon_x}(x) \cap S \subset A'$ contradiction

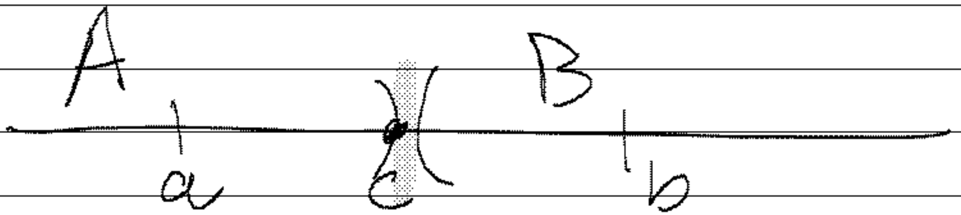
Because $y \in B'$ and $A' \cap B' = \emptyset$

<u>Prop</u>: Intervals in $\mathbb{R}$ are connected

<u>proof</u>: Let I be an interval in $\mathbb{R}$

Assume $I \subset A \cup B$ with $I \cap A \neq \emptyset$ $I \cap B \neq \emptyset$

$A \cap B = \emptyset$ and A and B open.



Let $a \in I \cap A$ and $b \in I \cap B$

Assume $a < b$.

Let $S = \{\underset{x \geq a}{x} : [a, x) \subset A\}$

note, $\exists \varepsilon > 0$ such that $a+\varepsilon \in S$

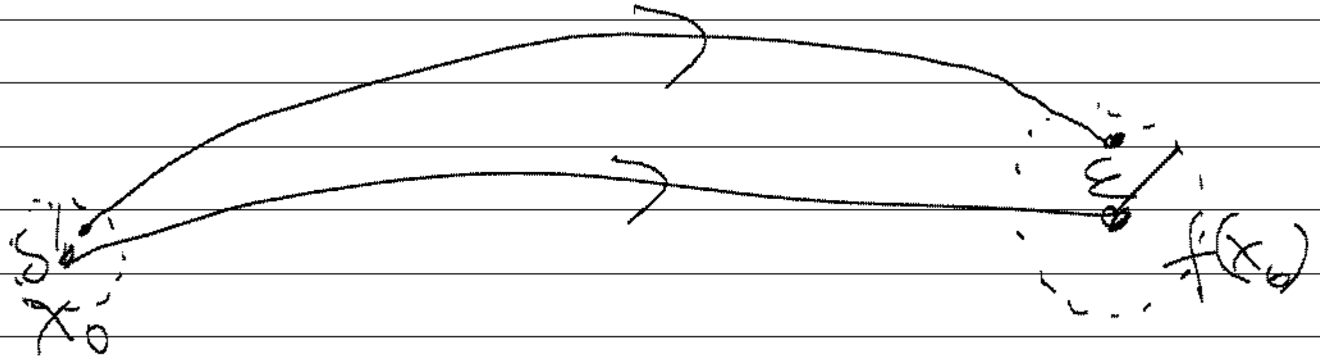
$S \neq \emptyset$ $a \in S$ $b \notin A$, thus, if $[a, x) \subset A \Rightarrow$
$x \leq b \Rightarrow b$ is a bound of S. Let $c = \text{lub}\, S$
Does $c \in A$? No. If $c \in A \Rightarrow \exists \varepsilon > 0$ such that $(-\varepsilon+c, \varepsilon+c)$
$\subset A \Rightarrow$ $\left.\begin{array}{l}[a, c) \subset A \text{ (you can prove)} \\ [c, c+\varepsilon) \subset A\end{array}\right\} \Rightarrow c+\varepsilon \in A$ contradiction,

$c \notin A$. Thus $c \in B \Rightarrow \exists \varepsilon > 0$: $(-\varepsilon+c, c+\varepsilon) \subset B$ contradiction,
because $c-\varepsilon$ would be an upper bound of S.

Continuous functions

Def: $f: E \to E'$. E and E' metric spaces with distances d and d' respectively. Let $x_0 \in E$.

f is continuous at x_0 if $\forall \varepsilon > 0 \ \exists \delta > 0 : d(x, x_0) < \delta \implies d'(f(x), f(x_0)) < \varepsilon$



<u>Def</u>: We say that f is continuous if it is continuous at every $x \in \overline{E}$