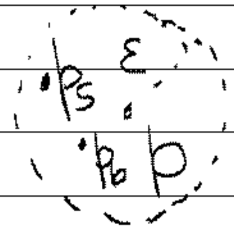
<u>Sequences</u> $p_n \in \mathbb{N}$ $\quad p_n \in E$

p is a limit of p_n if $\forall \varepsilon > 0$ $\exists N$ such that $n \geq N$

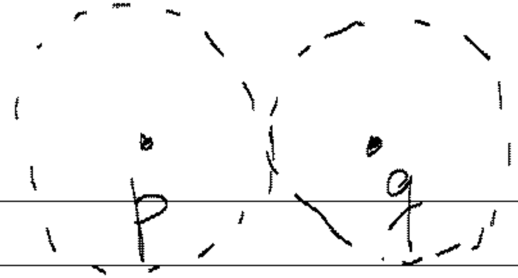
$$d(p_n, p) < \varepsilon$$



If p is a limit of p_n, we say the sequence converges to p

<u>Prop</u>: A sequence has at most one limit

<u>proof</u>: Assume both p and q are limits of p_n with $p \neq q$. Let $\varepsilon = \dfrac{d(p,q)}{2}$. $\exists N_1$ such that $d(p_n, p) < \varepsilon$



if $n \geq N_1$, $\exists N_2$ such that $d(p_n, q) < \varepsilon$ if $n \geq N_2$. This is because both p & q are limits of p_n. Take any $n \geq \max\{N_1, N_2\}$. Then

$$\left.\begin{array}{l} d(p_n, p) < \varepsilon \\ d(p_n, q) < \varepsilon \end{array}\right\} \Rightarrow d(p,q) \leq d(p, p_n) + d(p_n, q) < 2\varepsilon = d(p,q)$$

Contradiction.

<u>Notation</u>: $p = \lim_{n \to \infty} p_n$ or $p_n \longrightarrow p$

<u>Def</u>: n_i positive integers $n_{i+1} > n_i$ for all i

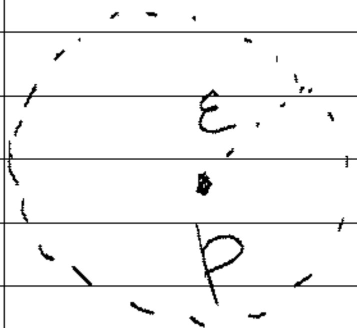
Let p_n be a sequence. p_{n_i} is called a subsequence of p_n.

Ex: $a_n = n^2$ $\qquad a_{2n} = (2n)^2$ $\qquad a_{2i} = (2i)^2$

Prop: $\lim_{n\to\infty} p_n = p \Longrightarrow \lim_{i\to\infty} p_{n_i} = p$

proof: Let $\varepsilon > 0$. Since $p_n \to p$ $\exists N$ such that $d(p_n, p) < \varepsilon$ if $n \geq N$. Let i_0 such that $n_{i_0} \geq N$. Then, if

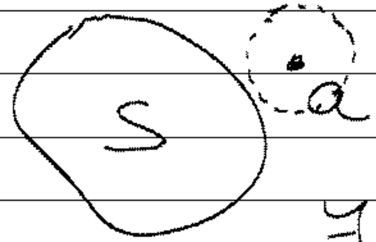
$i \geq i_0 \Rightarrow n_i \geq n_{i_0} \geq N \Rightarrow d(p_{n_i}, p) < \varepsilon$ ✓



Def: If p_n has a limit, we say that the sequence is convergent or converges.

Theorem: $S \subset E$. S is closed $\iff$ $\forall a_n \in S$ that converges, $\lim\limits_{n \to \infty} a_n \in S$

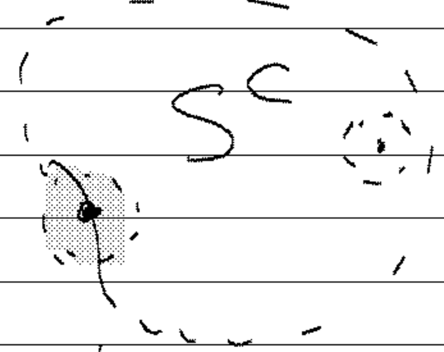
proof: $\Rightarrow$) S is closed. Let $a_n \in S$ $\forall n$. Assume $a_n \to a$.



If $a \notin S \Rightarrow a \in S^c$ which is open and thus, $\exists \varepsilon > 0$ such that $B_\varepsilon(a) \subset S^c$. But since $a_n \to a$ $\exists N$ such that $n \geq N \Rightarrow a_n \in B_\varepsilon(a) \subset S^c$ contradiction because $a_n \in S$ $\forall n$.

$\Leftarrow$) If S is not closed $\Rightarrow$ S^c is not open $\Rightarrow$ $\exists a \in S^c$

$B_\varepsilon(a) \not\subseteq S^c \quad \forall \varepsilon > 0$. Since $B_{1/n}(a) \cap S \neq \emptyset$, let



$a_n \in B_{1/n}(a) \cap S$. Then $a_n \to a$

$a_n \in S \quad \forall n$ but $a \in S^c$ contradiction

<u>Proposition</u>: $a_n, b_n \in \mathbb{R}$ $\quad \lim_{n\to\infty} b_n = b \quad \lim_{n\to\infty} a_n = a$

1) $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

2) $\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$

3) $\lim_{n\to\infty} a_n - b_n = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$

4) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $b \neq 0$

<u>proof</u>: $\left|\frac{a_n}{b_n} - \frac{a}{b}\right| = \frac{|a_n b - a b_n|}{|b b_n|} = \frac{|a_n b - ab + ab - ab_n|}{|b b_n|} \leq$

$\leq \frac{|a_n b - ab|}{|b b_n|} + \frac{|ab - ab_n|}{|b b_n|} = \frac{|b|\,|a_n - a| + |a|\,|b - b_n|}{|b b_n|}$ (1)

$b_n \to b$ $\quad |b| \leq |b - b_n| + |b_n| \Rightarrow \quad |b| - |b - b_n| \leq |b_n|$

$\exists N_1$ such that $n \geq N_1$ $\quad |b - b_n| < \frac{|b|}{2}$

then if $n \geq N_1$ $|b_n| \geq \frac{|b|}{2}$

$$1 \leq \frac{|b|\,|a_n - a| + |a|\,|b - b_n|}{\frac{|b|^2}{2}} \quad (*)$$

if $n \geq N_1$

$n \geq N_2 \Rightarrow |a_n - a| < \varepsilon \frac{|b|}{2}$

$n \geq N_3 \Rightarrow |b_n - b| < \frac{\varepsilon}{2} \frac{|b|^2}{2|a|}$ if $a \neq 0$, ($N_3 = 1$ if $a = 0$)

$\Rightarrow$ if $n \geq \max\{N_1, N_2, N_3\}$

$* < \varepsilon$