Note Title 3/2/2016

Ex: 1) Show $f(x) = x^2$ is continuous.

Let $x_0 \in \mathbb{R}$

$$d(f(x), f(x_0)) = |f(x) - f(x_0)| = |x^2 - x_0^2| = |x + x_0|\,|x - x_0|$$

Assume $|x - x_0| \leq 1 \Rightarrow |x + x_0| \leq 2|x_0| + 1$

Then $|f(x) - f(x_0)| \leq (2|x_0|+1)\,|x - x_0| < \varepsilon$

$$\delta = \min\left\{ \frac{\varepsilon}{2|x_0|+1}, 1 \right\}$$

Then, if $d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon$

<u>Ex 2</u>: $a \in E$ $\quad f: E \to \mathbb{R}$ $\quad f(x) = d(x, a)$

f is continuous

<u>proof</u>: Let $x_0 \in E$. Let $x \in E$

$$|f(x) - f(x_0)| = |d(x, a) - d(x_0, a)| \leq d(x, x_0) < \varepsilon$$

$$d(x, x_0) + d(x_0, a) \geq d(x, a) \implies d(x, x_0) \geq d(x, a) - d(x_0, a)$$

$$d(x, x_0) + d(x, a) \geq d(x_0, a) \implies d(x, x_0) \geq d(x_0, a) - d(x, a)$$

$\delta = \varepsilon$ if $d(x, x_0) < \delta \implies d(f(x), f(x_0)) \leq d(x, x_0) < \delta = \varepsilon$ ✓

Ex: $f: E \to E'$ $\quad x_0' \in E'$ $\quad f(x) = x_0'$ is continuous

Ex: $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$ is discontinuous at $x = 0$

Let $\varepsilon = \frac{1}{2}$. Let $\delta > 0$. Let $x = -\frac{\delta}{2}$ $\Rightarrow$ $d(x, 0) = \frac{\delta}{2} < \delta$

but $d(f(x), f(0)) = d(0, 1) = |0 - 1| = 1 > \varepsilon = \frac{1}{2}$. Thus f is discontinuous at $x = 0$.

Prop: $f: E \to E'$. f is continuous $\Leftrightarrow$ $\forall\, U' \subset E'$, U' open, $f^{-1}(U')$ is also open.

proof: $\Rightarrow$) Let $U' \subset E'$, U' open

$U = f^{-1}(U') = \{x \in E : f(x) \in U'\}$

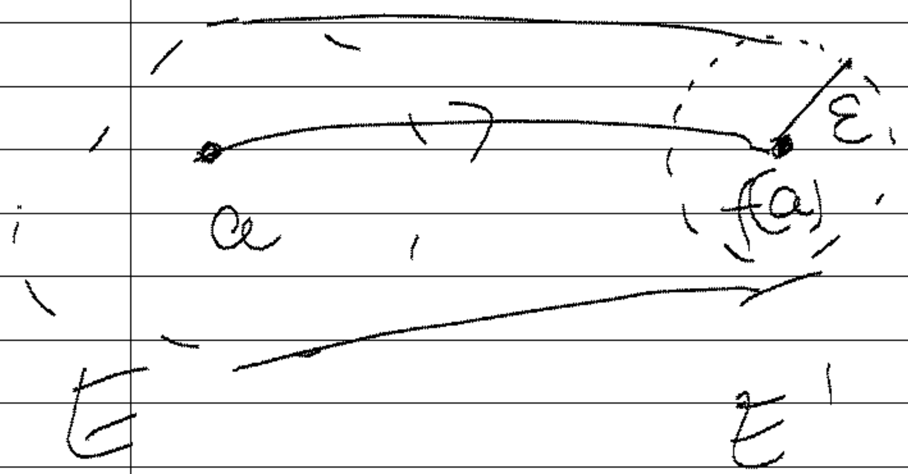
Let $a \in U$. Since $a \in U = f^{-1}(U')$ then $f(a) \in U'$. Since U' is

open, $\exists\ \varepsilon > 0$ such that $B_\varepsilon(f(a)) \subset U'$. Since f is continuous at a $\exists\ \delta > 0$ such that $d(x,a) < \delta \Rightarrow$

$d(f(x), f(a)) < \varepsilon \Rightarrow f(x) \in B_\varepsilon(f(a)) \subset U' \Rightarrow x \in U = f^{-1}(U')$

$\Rightarrow B_\delta(a) \subset U \Rightarrow U$ is open.

($\Leftarrow$) Let $a \in E$. Let $\varepsilon > 0$. $B_\varepsilon(f(a))$ is open, thus



$f^{-1}(B_\varepsilon(f(a)))$ is open in E

Since $a \in f^{-1}(B_\varepsilon(f(a))) \Rightarrow \exists\, \delta > 0$ such that $B_\delta(a) \subset f^{-1}(B_\varepsilon(f(a)))$

This means that $d(x,a) < \delta \Rightarrow d'(f(x), f(a)) < \varepsilon$

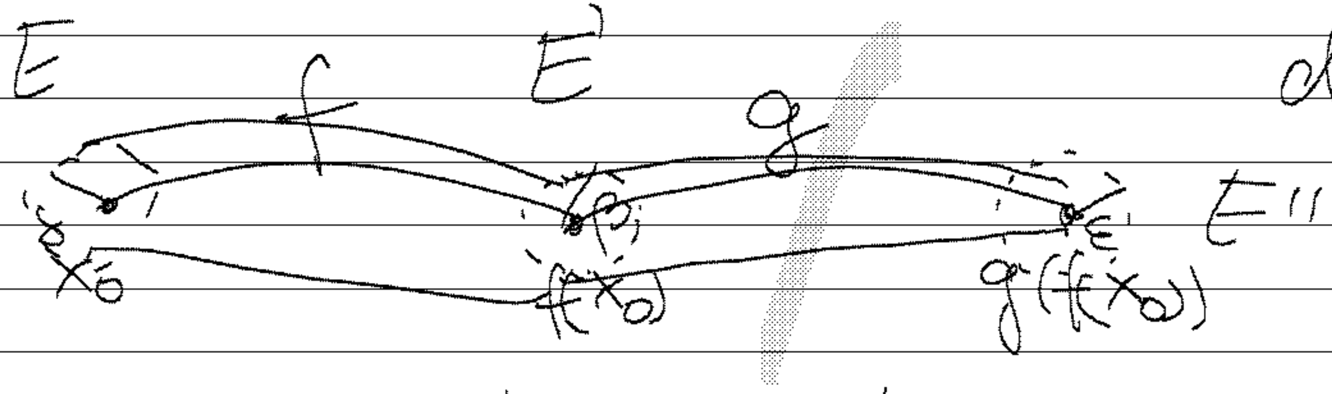
<u>Corollary</u>: $f: E \to \mathbb{R}$ continuous. $a \in \mathbb{R} \Rightarrow$
$\{x \in E : f(x) > a\}$ and $\{x \in E : f(x) < a\}$ are both open

$\quad\quad\quad\quad \| \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \|$

$f^{-1}((a,\infty)) \quad\quad\quad\quad f^{-1}((-\infty,a))$

<u>Prop</u>: $f: E \to E'$ $\quad g: E' \to E''$ $\quad x_0 \in E$. f continuous at x_0, g continuous at $f(x_0) \Rightarrow g \circ f$ is continuous at x_0

<u>proof</u>: Let $\varepsilon > 0$. Since g is continuous at $f(x_0)$, $\exists\, \beta > 0$ such that $d(y, f(x_0)) < \beta \Rightarrow$ $d(g(y), g(f(x_0))) < \varepsilon$



Since f is continuous at x_0, $\exists\, \delta > 0$ such that $d(x, x_0) < \delta$

$\Rightarrow \quad d(f(x), f(x_0)) < \beta \Rightarrow \quad d(g(f(x)), g(f(x_0))) < \varepsilon$

Thus, $g \circ f$ is continuous at x_0.

<u>Limits</u>:

~~Def~~: $x_0 \in E$. $\quad f: E \to E'$

or $f: E - \{x_0\} \to E'$

We say $\lim_{x \to x_0} f(x) = q$ if the function $\bar{f}(x) = \begin{cases} f(x) & \text{if } x \neq x_0 \\ q & \text{if } x = x_0 \end{cases}$

is continuous at x_0.