Note Title 3/11/2016

Def: $f_n: E \to E'$ $n \in \mathbb{N}$. $f: E \to E'$. We say

1) f_n converges at x if $\lim_{n\to\infty} f_n(x)$ exists

Let $x \in E$.

2) f_n converges if f_n converges $\forall x \in E$.

In this case, $f = \lim_{n\to\infty} f_n$ if $f(x) = \lim_{n\to\infty} f_n(x)$ $\forall x \in E$

Examples: 1) $\lim_{n\to\infty} \underbrace{x - \frac{x}{n}}_{f_n(x)} = \underbrace{x}_{f(x)}$

$x \in [0,1]$ $\quad f_n(x) = x^n \quad f_n : [0,1] \to \mathbb{R}$

2) $\lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$

$f_n : E \to E'$

Def: $f_n : E \to E'$. E & E' both metric. $f : E \to E'$

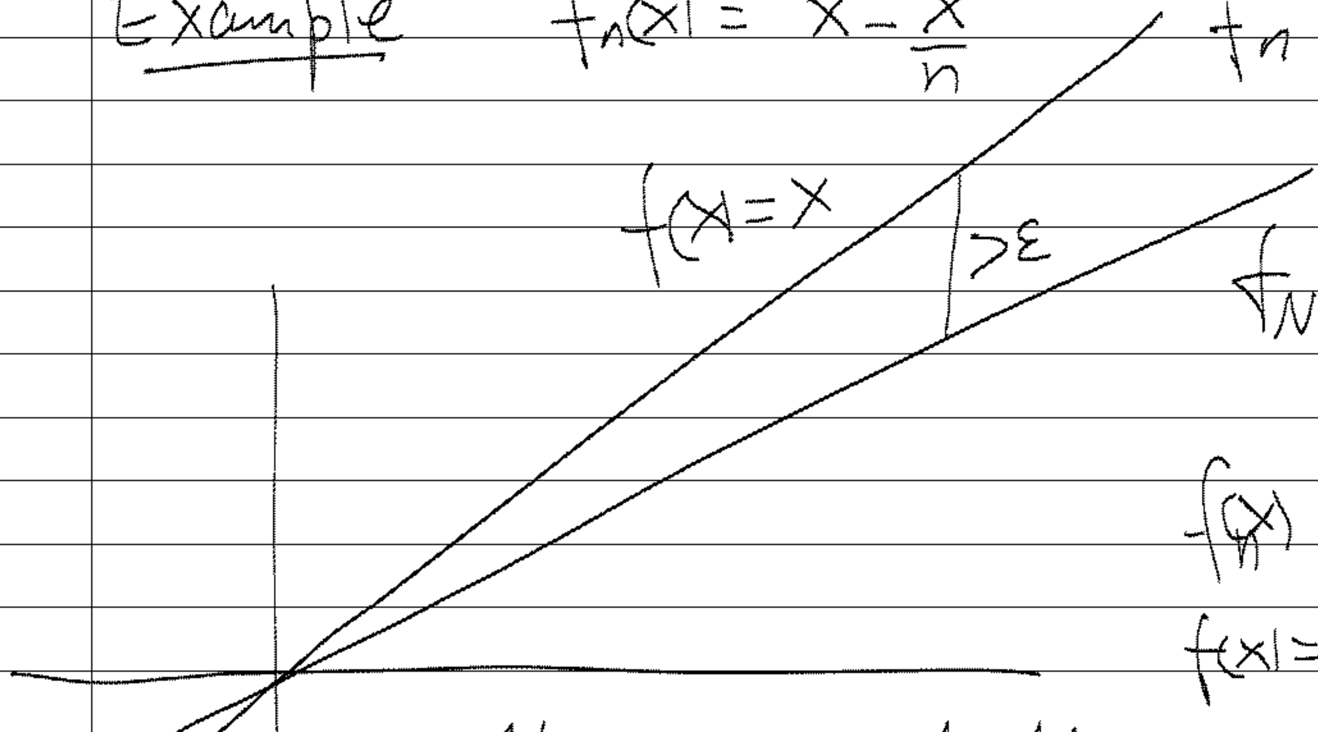
f_n converges uniformly to f if $\forall \varepsilon > 0$ $\exists N$ such that $d(f_n(x), f(x)) < \varepsilon$
if $n \geq N$ $\forall x \in E$ $\qquad \underset{N_\varepsilon}{\overset{\shortparallel}{N}}$

Compare with

f_n converges to f if $\forall \varepsilon > 0$ $\forall x \in E$ $\exists N = N_{x\varepsilon}$ such that $n \geq N$

$d(f_n(x), f(x)) < \varepsilon$

<u>Example</u> $f_n(x) = x - \frac{x}{n}$ $\quad f_n : \mathbb{R} \to \mathbb{R}$



$f_N(x) = x - \frac{x}{N}$

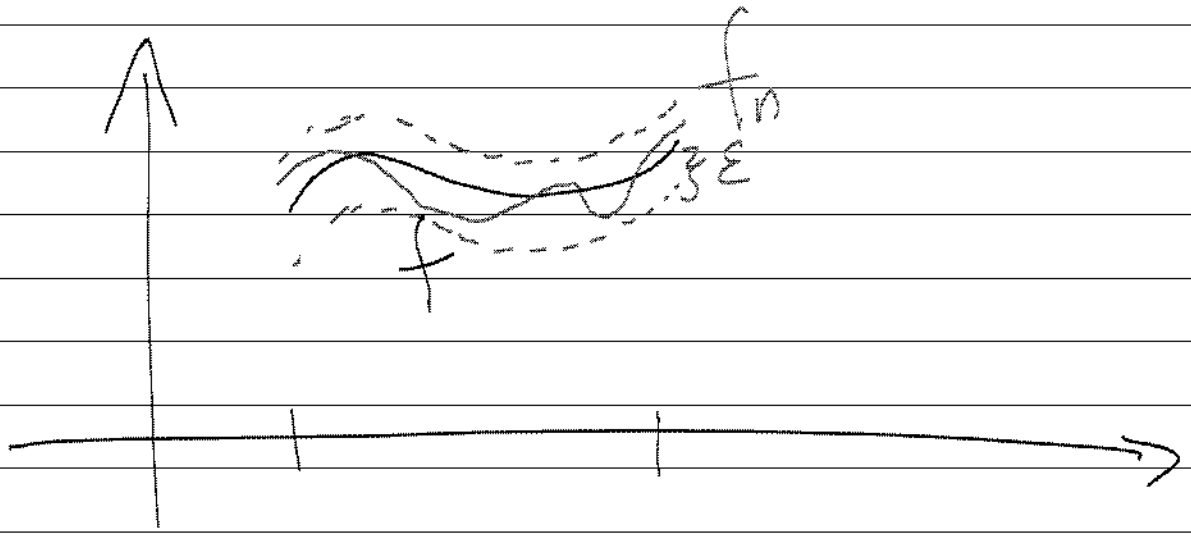
$f_n(x)$ does not uniformly converge to $f(x) = x$ because:

Let $\varepsilon > 0$ and $N \in \mathbb{N}$, I need to find $x \in E$ and

$n \geq N$ such that $d(f_n(x), f(x)) \geq \varepsilon$

$|f_N(x) - f(x)| = \frac{|x|}{N}$ if $|x| \geq N\varepsilon \Rightarrow d(f_N(x), f(x)) \geq \varepsilon$ ✓

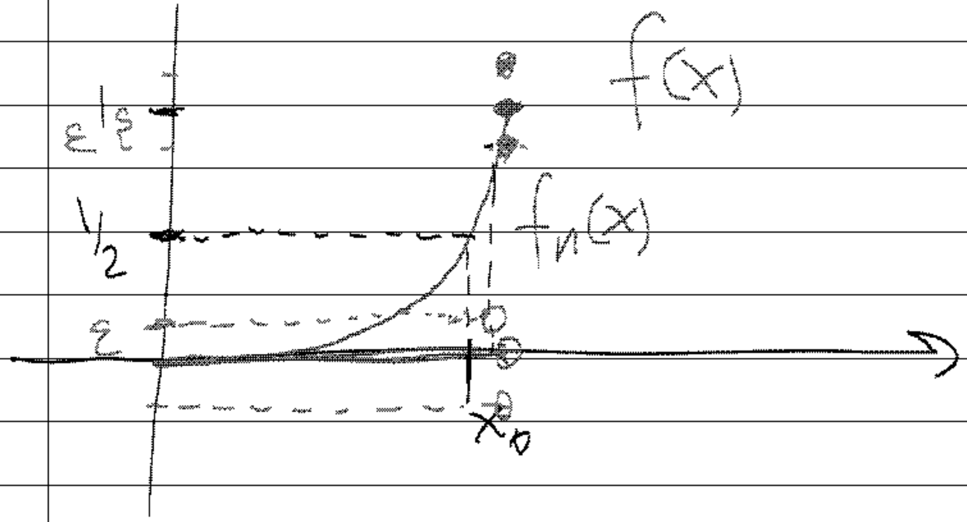
$\overset{''}{d}(f_N(x), f(x))$



$\underline{Ex}$: $f_n(x) = x^n$ $\quad f:[0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

$x_0^n = \frac{1}{2}$ $\quad x_0 = \left(\frac{1}{2}\right)^{1/n}$



$d(f_n(x_0), f(x)) = \frac{1}{2}$

Obs: $f_n(x) \to f(x)$ uniformly $\Longrightarrow$ $f_n(x) \to f(x)$ pointwise

Prop: $f_n : E \to E'$. E' complete. Then:

f_n converges uniformly $\Longleftrightarrow$ $\forall \varepsilon > 0$ $\exists N$ such that if $n, m \geq N$

$d(f_n(x), f_m(x)) < \varepsilon \quad \forall x \in E$

proof: $\Rightarrow$) $f_n \to f$ uniformly. Let $\varepsilon > 0$. $\exists N$: $d(f_n(x), f(x)) < \frac{\varepsilon}{2}$ $\forall x$ if $n \geq N$. Then, if $n, m \geq N$

$$\left.\begin{aligned} d(f_n(x), f(x)) &< \frac{\varepsilon}{2} \\ d(f_m(x), f(x)) &< \frac{\varepsilon}{2} \end{aligned}\right\} \forall x \in E$$

$$d(f_n(x), f_m(x)) \leq d(f_n(x), f(x)) + d(f_m(x), f(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\Leftarrow$) For each x, $f_n(x)$ is a Cauchy sequence in E'. Since E' is complete, $f_n(x)$ converges. Call this limit $f(x) = \lim_{n\to\infty} f_n(x)$

Let $\varepsilon > 0$ $\exists N$ such $n, m \geq N$ $d(f_n(x), f_m(x)) < \varepsilon$ $\forall x$

Let $n \geq N$, x fixed

$$d(f_n(x), f(x)) \leq \underbrace{d(f_n(x), f_m(x))} + \underbrace{d(f_m(x), f(x))}$$

now take $m \geq N$ to get

$$d(f_n(x), f(x)) < \varepsilon + \underbrace{d(f_m(x), f(x))}_{\to 0}$$ now take lim $m \to \infty$

$1 < 1 + \varepsilon$

$$d(f_n(x), f(x)) \leq \varepsilon \quad \forall x \quad \forall n \geq N$$

$1 \leq 1$