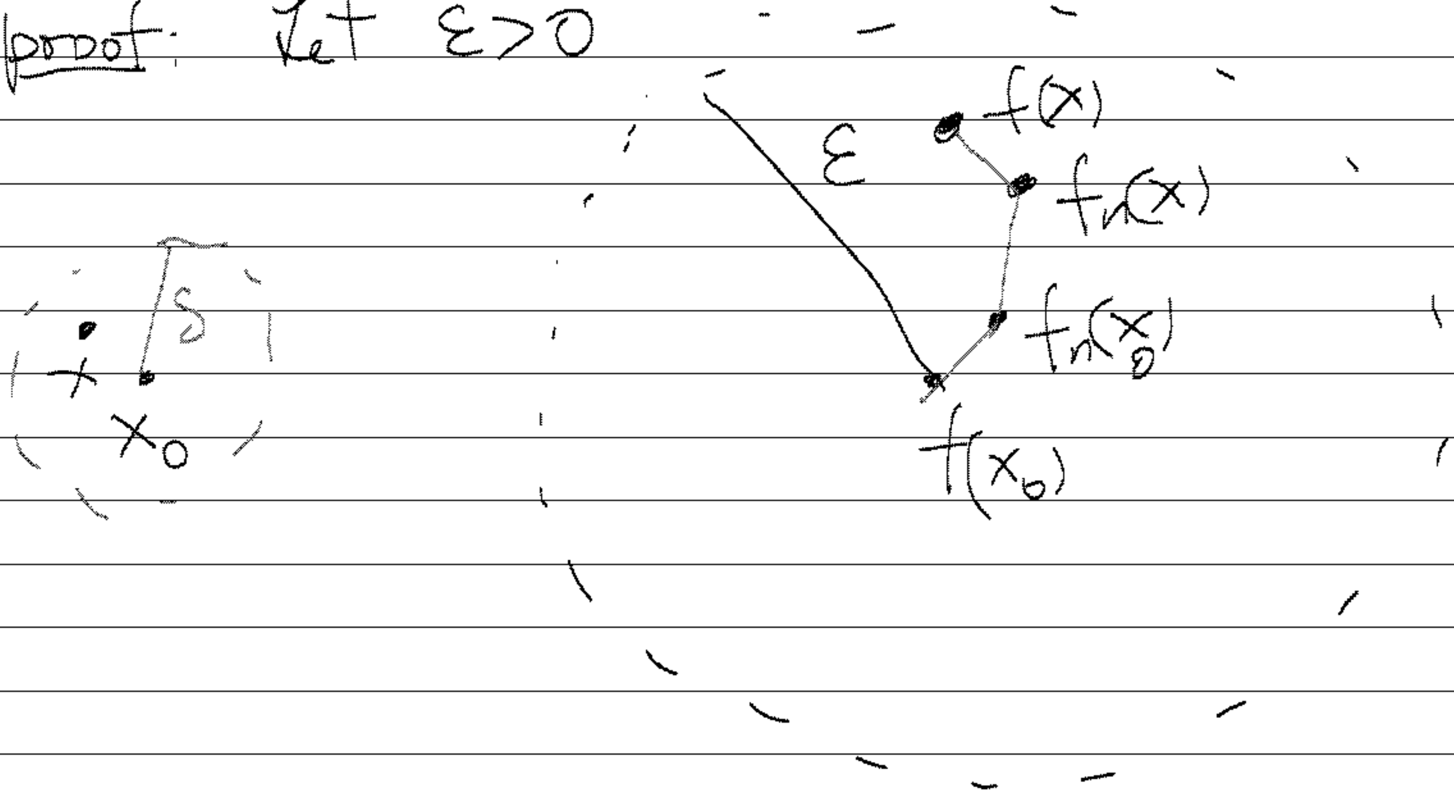
Note Title 3/14/2016

Theorem: $f_n \to f$ uniformly. f_n are continuous $\Longrightarrow$ f is continuous

proof: Let $\varepsilon > 0$



(1) $d(f(x), f(x_0)) \leq d(f(x), f_n(x)) + d(f_n(x), f_n(x_0)) + d(f_n(x_0), f(x_0))$

1st pick n such that

$d(f_n(y), f(y)) < \frac{\varepsilon}{3}$ $\forall y$ I can do that because of uniform convergence

Once I have n, pick δ such that $d(x, x_0) < \delta \Rightarrow$

$d(f_n(x), f_n(x_0)) < \frac{\varepsilon}{3}$. I can do that because f_n is continuous

then $d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon$ from Eq (1)

Lemma: $f: E \to E'$ $g: E \to E'$ both continuous

$h: E \to \mathbb{R}$ $h(x) = d(f(x), g(x))$. then h is continuous

Proof: Let $x_0 \in E$. Let $\varepsilon > 0$

$$d(h(x), h(x_0)) = |h(x) - h(x_0)| = |d(f(x), g(x)) - d(f(x_0), g(x_0))|$$

Let $\delta > 0$ such that $d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \frac{\varepsilon}{2}$

and $d(g(x), g(x_0)) < \frac{\varepsilon}{2}$

$$d(f(x), g(x)) \leq d(f(x), f(x_0)) + d(f(x_0), g(x_0)) + d(g(x_0), g(x))$$

$$< d(f(x_0), g(x_0)) + \varepsilon$$

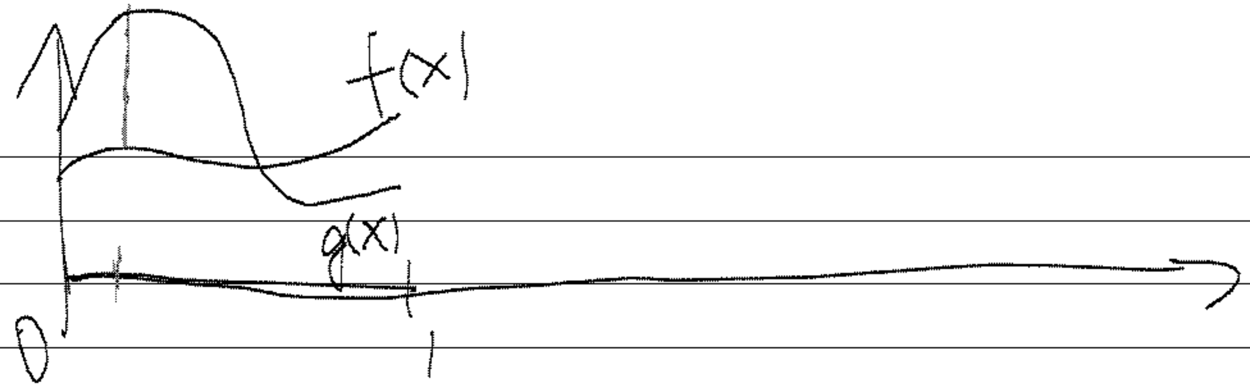
$d(f(x), g(x)) - d(f(x_0), g(x_0)) < \varepsilon$ Exchange the roles of x and x_0 to get

$|d(f(x), g(x)) - d(f(x_0), g(x_0))| < \varepsilon$ ✓

<u>Def</u>: E compact. E' metric $\mathcal{F} = \{f : E \to E' \text{ continuous}\}$

For $f, g \in \mathcal{F}$ we define

$$D(f,g) = \max_{x \in E} \underbrace{d(g(x), f(x))}_{= h(x) \text{ is continuous}}$$



<u>Obs</u>: $D(f,g)$ is well defined.

<u>Obs</u>: D is a distance on $\mathcal{F}$. Thus, $(\mathcal{F}, D)$ is a metric space.

Check Δ inequality. $f, g, u \in \mathcal{F}$ $\quad x \in E$

$$d(f(x), g(x)) \leq d(f(x), u(x)) + d(u(x), g(x)) \leq D(f,u) + D(u,g)$$

take maximum on the left hand side to get

$D(f,g) \leq D(f,u) + D(u,g)$

Obs: E compact. $f_n \to f$ in $(\mathcal{F}, D) \iff f_n \to f$ uniformly.

proof: $\Rightarrow$) $f_n \to f$ in $(\mathcal{F}, D)$. Let $\varepsilon > 0$. Then, $\exists N: n \geq N$

$D(f_n, f) < \varepsilon \Rightarrow d(f_n(x), f(x)) \leq \max_{y \in E} d(f_n(y), f(y)) = D(f_n, f) < \varepsilon$

let $x \in E$ if $n \geq N$

$\Leftarrow$) $f_n \to f$ uniformly. Let $\varepsilon > 0 \Rightarrow \exists N: n \geq N$ $d(f_n(x), f(x)) < \varepsilon$

$\forall x \in E$. Take max over all $x \in E$ to get

$\max_{x \in E} d(f_n(x), f(x)) = D(f_n, f) < \varepsilon \Rightarrow f_n \to f$ in $(\mathcal{F}, D)$

<u>Theorem</u>: E compact, E' complete. Then $(\mathcal{F}, D)$ is complete

<u>proof</u>: Let f_n be a Cauchy sequence. Then $\forall \varepsilon > 0$
$\exists N$ such that $n, m \geq N \Rightarrow D(f_n, f_m) < \varepsilon$

Let $x \in E$. If $n, m \geq N \Rightarrow d(f_n(x), f_m(x)) \leq D(f_n, f_m) < \varepsilon$
$\Rightarrow f_n(x)$ is Cauchy. Since E' is complete, $f_n(x)$ converges.
Let $f(x) = \lim_{n \to \infty} f_n(x)$

If $n \geq N$ $\quad d(f_n(x), f(x)) \leq \underbrace{d(f_n(x), f_m(x))}_{< \varepsilon} + \underbrace{d(f_m(x), f(x))}_{\longrightarrow 0}$

take $m \to \infty$

$$d(f_n(x), f(x)) \leq \varepsilon \quad \forall\, x \in E$$

take maximum

$$D(f_n, f) \leq \varepsilon \implies f_n \to f \text{ in } (\mathcal{F}, D)$$