<u>Def</u>: $f : U \to \mathbb{R}$ U open $U \subset \mathbb{R}$ $x_0 \in U$. f is differentiable at x_0 if $\lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$ exists. If this is the case, we set $f'(x_0) = \lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$, this is the derivative of f at x_0.

<u>Obs</u>: $f'(x_0)$ exists $\iff$ $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $0 < |x - x_0| < \delta$ $\Rightarrow \left| \dfrac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \varepsilon$ $\iff$ $\forall \varepsilon > 0$ $\exists \delta > 0$ such that

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - f(x_0) - f'(x_0)(x - x_0)| < \varepsilon |x - x_0|$$

Prop: $U \subset \mathbb{R}$, U open, $f: U \to \mathbb{R}$ $x_0 \in U$. If f is differentiable at x_0, then f is continuous at x_0.

Proof: $f'(x_0)$ exists. $\Rightarrow$ $\forall \beta > 0$ $\exists \delta_1 > 0$ such that

$$0 < |x - x_0| < \delta_1 \Rightarrow |f(x) - f(x_0) - f'(x_0)(x - x_0)| < \beta\,|x - x_0|$$

then

$$|f(x) - f(x_0)| \leq |f(x) - f(x_0) - f'(x_0)(x - x_0) + f'(x_0)(x - x_0)| \leq$$

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| + |f'(x_0)|\,|x - x_0| < (\beta + |f'(x_0)|)\,|x - x_0| < \varepsilon$$

then, take $|x-x_0| < \min\left\{\delta_1, \frac{\varepsilon}{\beta+|f'(x_0)|}\right\} = \delta$

<u>Def</u>: $U \subset \mathbb{R}$, U open, $f: U \to \mathbb{R}$
f is differentiable on U if it is differentiable at x $\forall x \in U$.

<u>Notation</u>: $f' = \frac{df}{dx}$

<u>Example</u>: 1) $f(x) = |x|$ is continuous at $x=0$ but not differentiable at $x=0$

$$\lim_{x\to 0^+} \frac{|x|-|0|}{x-0} = 1 \qquad \lim_{x\to 0^-} \frac{|x|-|0|}{x-0} = \lim_{x\to 0^-} \frac{-x}{x} = -1$$

2) $f(x) = k$ $\qquad f'(x) = 0 = \lim_{y\to x} \frac{f(y)-f(x)}{y-x} = \lim_{y\to x} \frac{k-k}{y-x} = 0$

3) $\frac{d}{dx} x = 1$

Prop: $f, g : U \to \mathbb{R}$ $\quad U \subset \mathbb{R}$ open $\quad f$ & g differentiable at $x_0 \in U$. then

1) $(f+g)'(x_0) = f'(x_0) + g'(x_0)$

2) $(f-g)'(x_0) = f'(x_0) - g'(x_0)$

3) $(cf)'(x_0) = c f'(x_0)$

4) $(fg)'(x_0) = f'(x_0)\,g(x_0) + f(x_0)\,g'(x_0)$

5) $\left(\dfrac{f}{g}\right)'(x_0) = \dfrac{f'(x_0)\,g(x_0) - f(x_0)\,g'(x_0)}{(g(x_0))^2}$ if $g(x_0) \neq 0$.

~~proof~~. $\displaystyle\lim_{x\to x_0} \frac{(fg)(x) - (fg)(x_0)}{x - x_0} = \lim_{x\to x_0} \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0} = \lim_{x\to x_0} g(x)\left(\frac{f(x) - f(x_0)}{x - x_0}\right) + \lim_{x\to x_0} f(x_0)$

$$\overrightarrow{g(x_0)} \qquad \overrightarrow{f'(x_0)}$$

$$\left(\underbrace{\frac{g(x)-g(x_0)}{x-x_0}}_{\to\, g'(x_0)}\right) = f'(x_0)\,g(x_0) + f(x_0)\,g'(x_0)$$

<u>Corollary</u>: n integer, positive $\Rightarrow \frac{d}{dx} x^n = n x^{n-1}$

<u>proof</u> by induction. $n=1$ ✓

$n>1$. $x^n = x^{n-1} x$ use product rule to get

$$\frac{d}{dx} x^n = \frac{d}{dx} x^{n-1} x = (n-1) x^{n-2} x + x^{n-1} (1) = n x^{n-1}$$

<u>Prop</u>: $f: U \to V$ $\quad g: V \to \mathbb{R}$ $\quad U, V$ open in $\mathbb{R}$
$x_0 \in U$ f differentiable at x_0 and g differentiable at $f(x_0)$. Then $g \circ f$ is differentiable at x_0 and

$$(g \circ f)'(x_0) = g'(f(x_0))\, f'(x_0)$$

<u>proof</u>: Let $y_0 = f(x_0)$. Let $A: V \to \mathbb{R}$ $\quad A(y) = \begin{cases} \dfrac{g(y) - g(y_0)}{y - y_0} & \text{if } y \neq y_0 \\ g'(y_0) & \text{if } y = y_0 \end{cases}$

A is continuous at y_0

$$\lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} = \lim_{x \to x_0} A(f(x)) \cdot \left(\frac{f(x) - f(x_0)}{x - x_0} \right) =$$

$$A(f(x_0))\, f'(x_0) = g'(f(x_0))\, f'(x_0)$$

Prop: $f: U \to \mathbb{R}$ $\quad U \subset \mathbb{R}$ $\quad U$ open

f attains a maximum (local) at $x_0 \in U$. then, if $f'(x_0)$ exists, $f'(x_0) = 0$

proof: $$\lim_{x \to x_0^+} \frac{\overbrace{f(x) - f(x_0)}^{\leq 0}}{\underbrace{x - x_0}_{> 0}} \leq 0$$

$$\Rightarrow f'(x_0) = 0$$

$$\lim_{x \to x_0} \frac{\overbrace{f(x) - f(x_0)}^{\leq 0}}{\underbrace{x - x_0}_{< 0}} \geq 0$$