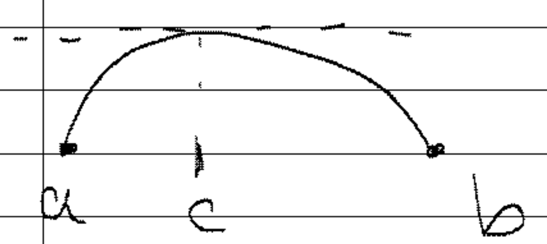
Note Title 4/4/2016

Rolle's Lemma. $f:[a,b]\longrightarrow\mathbb{R}$ continuous and differentiable on (a,b) $f(a)=f(b)=0$. then $\exists\, c\in(a,b)$ such that $f'(c)=0$

proof: Let $M=\max\limits_{x\in[a,b]} f(x)$. If $M>0$. Let c such that



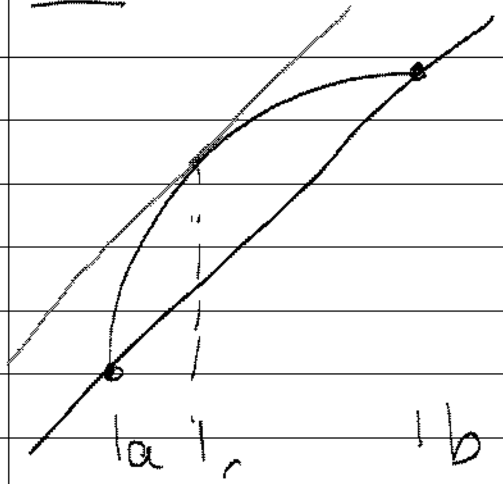
$f(c)=M$. $c\in(a,b)\Rightarrow f'(c)=0$

If $M=0$. Let $m=\min\limits_{x\in[a,b]} f(x)$. If $m<0$,

similar arguments....... $f'(c)=0$. If $m=M=0\Rightarrow f=0\ \forall x$

$\in (a,b) \Rightarrow f'(c) = 0 \quad \forall x \in (a,b)$

<u>Th</u>: (MVT) $f: [a,b] \to \mathbb{R}$ continuous & differentiable in (a,b). then $\exists\, c \in (a,b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$



Proof: $l(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$

$g(x) = f(x) - l(x)$ then $g(a) = g(b) = 0$

$\Rightarrow \exists\, c \in (a,b)$ such that $g'(c) = 0$.

$g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$ ✓

Corollary: $f:(a,b) \to \mathbb{R}$ differentiable and $f'(x)=0$ $\forall\, x \in (a,b)$. Then $f(x)=k$ $\forall\, x \in (a,b)$ and some constant k.

proof: Let $a<x<y<b$. From MVT $\exists\, z$: $x<z<y$ and $\underset{0}{\underset{\|}{f'(z)}} = \dfrac{f(y)-f(x)}{y-x} \Rightarrow f(y)=f(x)$ ✓

Corollary: $f,g:(a,b) \to \mathbb{R}$ both differentiable and $f'(x)=g'(x)$ $\forall\, x \in (a,b) \Rightarrow \exists\, K \in \mathbb{R}$ such that $g(x)=f(x)+K$

proof: $h(x)=f(x)-g(x)$ $h'(x)=0$ $\forall\, x \Rightarrow h(x)=-K$ for some $K \in \mathbb{R}$.

Def: $f: U \to \mathbb{R}$, $U \subseteq \mathbb{R}$, f is called:

1) increasing if $a < b$, $a, b \in U \Rightarrow f(a) \leq f(b)$
2) strictly increasing if $a, b \in U$, $a < b \Rightarrow f(a) < f(b)$
3) Similar for decreasing.

Corollary: $f: (a,b) \to \mathbb{R}$ differentiable. $f'(x) \geq 0 \ \forall x \in (a,b)$
$\Rightarrow f$ is increasing

pf: $a < x < y < b$, then $\exists \ c \in (x,y)$: $\underset{\geq 0}{f'(c)} = \dfrac{f(y) - f(x)}{y - x}$

then $f(y) - f(x) \geq 0$ ✓

Similar statement with $f'(x) \leq 0$ & f decreasing

Taylor

Def: $U \subset \mathbb{R}$, U open, $f: U \to \mathbb{R}$

1) f is n times differentiable if $f^{(n-1)}: U \to \mathbb{R}$ is differentiable. We set $f^{(n)} = (f^{(n-1)})'$ and $f^{(0)} = f$

2) $x_0 \in U$. f is n times differentiable at x_0 if $\exists V$, neighborhood of x_0 such that f is $n-1$ times differentiable at V and $f^{(n-1)}$ is differentiable at x_0

Lemma: U open interval. $f: U \to \mathbb{R}$ $n+1$ times differentiable $a, b \in U$. Define $R_n(b,a)$ as

$$f(b) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(b-a)^i + R_n(b,a)$$

then $\frac{d}{dx} R_n(b,x) = -\frac{f^{(n+1)}(x)}{n!}(b-x)^n$ for all $x \in U$.

Proof: $f(b) = \sum_{i=0}^{n} \frac{f^{(i)}(x)}{i!}(b-x)^i + R_n(b,x)$

$$0 = \sum_{i=1}^{n} \underbrace{\left[\frac{f^{(i+1)}(x)}{i!}(b-x)^{i} - \frac{f^{(i)}(x)}{(i-1)!}(b-x)^{i-1}\right]} + f'(x) + \frac{d}{dx}R_n(b,x)$$

$$0 = \frac{f^{(n+1)}(x)}{n!}(b-x)^{n} - f'(x) + f'(x) + \frac{d}{dx}R_n(b,x)$$

$$\frac{d}{dx}R_n(b,x) = -\frac{f^{(n+1)}(x)}{n!}(b-x)^{n}$$

<u>Taylor thm</u>: U open interval in $\mathbb{R}$ $\quad f: U \to \mathbb{R}$

f $(n+1)$ times differentiable $a, b \in U$ then

$$f(b) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

for some c between a and b.

<u>proof</u>: $R_n(b,a) = f(b) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k$

$$\frac{d}{dx} R_n(b,x) = -\frac{f^{(n+1)}(x)}{n!}(b-x)^n$$

Let $K = \frac{(n+1)!}{(b-a)^{n+1}} R_n(b,a)$

$\varphi : U \to \mathbb{R}$ $\qquad \varphi(x) = R_n(b,x) - \frac{K(b-x)^{n+1}}{(n+1)!}$

$\varphi(b) = R_n(b,b) - 0 = 0$

$\varphi(a) = R_n(b,a) - \frac{(n+1)!\, R_n(b,a)}{(b-a)^{n+1}} \frac{(b-a)^{n+1}}{(n+1)!} = 0$

Then, $\exists c$ between a and b such that $\varphi'(c) = 0$

$$\varphi'(x) = \frac{d}{dx} R_n(b,x) + \frac{K(b-x)^n}{n!} = -\frac{f^{(n+1)}(x)(b-x)^n}{n!} - \frac{(n+1)!}{(b-a)^{n+1}} R_n(b,a) \frac{(b-x)^n}{n!}$$

Set $x = c$

$$0 = -\frac{f^{(n+1)}(c)}{n!}(b-c)^n + \frac{(n+1)!}{(b-a)^{n+1}} R_n(b,a) \frac{(b-c)^n}{n!}$$

$$R_n(b,a) = \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$ ✓