Note Title 3/4/2016

Obs: $\lim_{x \to x_0} f(x) = q \iff \forall \varepsilon > 0 \ \exists \delta > 0$ such that

$0 < d(x, x_0) < \delta \implies d(f(x), q) < \varepsilon$

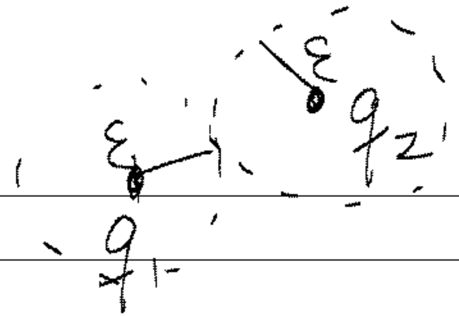
Obs: If x_0 is a cluster point of E and $f: E \to E'$

or $f: E - \{x_0\} \to E'$

$\lim_{x \to x_0} f(x)$ is unique if it exists

proof: Assume $q_1 = \lim_{x \to x_0} f(x)$ and $q_2 = \lim_{x \to x_0} f(x)$. Assume $q_1 \neq q_2$

Let $\varepsilon = d(q_1, q_2)/2$, Note $\varepsilon > 0$



$\exists\, \delta_1$ such that $0 < d(x, x_0) < \delta_1 \Rightarrow d(f(x), q_1) < \varepsilon$

$\exists\, \delta_2$ such that $0 < d(x, x_0) < \delta_2 \Rightarrow d(f(x), q_2) < \varepsilon$

Let x be such that $0 < d(x, x_0) < \min\{\delta_1, \delta_2\}$. Such x exists because x_0 is a cluster point of E. Then

$$2\varepsilon = d(q_1, q_2) \le \underbrace{d(q_1, f(x))}_{<\varepsilon} + \underbrace{d(f(x), q_2)}_{<\varepsilon} < 2\varepsilon \quad \text{contradiction}$$

because $d(x, x_0) < \min\{\delta_1, \delta_2\}$

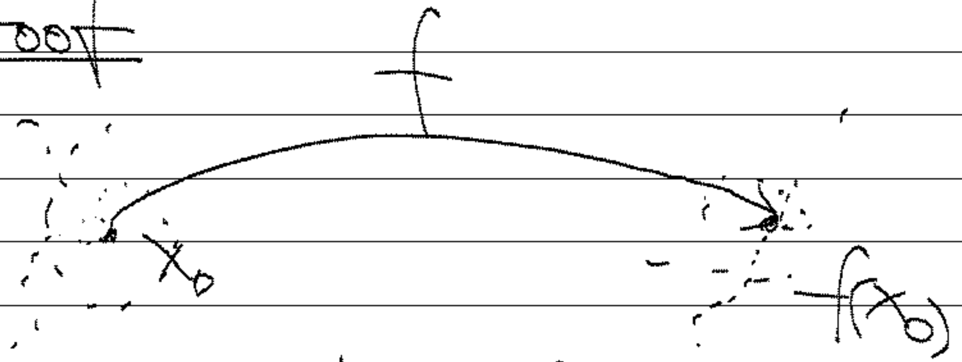
<u>Notation</u>: $S \subset E$ $\quad x_0$ a cluster point of S

$\lim\limits_{\substack{x \to x_0 \\ x \in S}} f(x)$ <u>Ex</u>: $\lim\limits_{x \to x_0^+} f(x) = \lim\limits_{\substack{x \to x_0 \\ x > x_0}} f(x)$

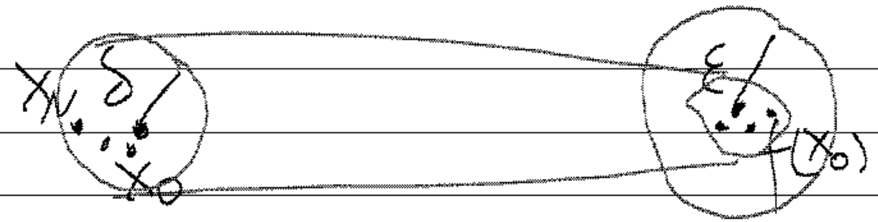
$E = \mathbb{R}$

<u>Proposition</u>: $f: E \to E'$ is continuous at $x_0 \iff \forall\, x_n$ sequence in E such that $x_n \to x_0$ we have $f(x_n) \to f(x_0)$

<u>proof</u>



$\Rightarrow$) Let x_n such that $x_n \to x_0$. Let $\varepsilon > 0$. Since f is continuous at x_0, $\exists\, \delta > 0$ such that $d(x, x_0) < \delta$ implies $d(f(x), f(x_0)) < \varepsilon$. Since $x_n \to x_0$ $\exists\, N$ such that



$n \geq N \quad d(x_n, x_0) < \delta \Rightarrow$

$d(f(x_n), f(x_0)) < \varepsilon \Rightarrow f(x_n) \to f(x_0)$

$\Leftarrow$) We need to show f is continuous at x_0

By contradiction. Assume f is not continuous at x_0. We need to find $x_n \to x_0$ but $f(x_n) \not\to f(x_0)$

$$\boxed{\begin{array}{c} A \Rightarrow B \\ \hline \text{not } B \Rightarrow \text{not } A \end{array}}$$

Since f is not continuous at x_0 $\exists\, \varepsilon > 0$ such that $\forall\, \delta > 0$ $\exists\, x$ such that $d(x, x_0) < \delta$ but $d(f(x), f(x_0)) > \varepsilon$

Let x_n be such that $d(x_n, x_0) < \frac{1}{n}$ but $d(f(x_n), f(x_0)) > \varepsilon$

Note $x_n \to x_0$ but $f(x_n) \not\to f(x_0)$

<u>Proposition</u>: $f: E \to \mathbb{R}$ and $g: E \to \mathbb{R}$ both continuous at x_0. Then $f+g$, $f \cdot g$ and $\frac{f}{g}$ (if $g(x_0) \neq 0$) are all continuous at x_0.

let $\varepsilon > 0$

proof: $d((f+g)(x), (f+g)(x_0)) = |f(x)+g(x)-(f(x_0)+g(x_0))| \leq$

$\leq |f(x)-f(x_0)| + |g(x)-g(x_0)| < \varepsilon$ if $d(x, x_0) < \delta = \min\{\delta_1, \delta_2\}$

Since f is cont at x_0 $\exists\, \delta_1$: $d(x, x_0) < \delta_1 \Rightarrow |f(x)-f(x_0)| < \frac{\varepsilon}{2}$

Since g is cont at x_0 $\exists\, \delta_2$: $d(x, x_0) < \delta_2 \Rightarrow |g(x)-g(x_0)| < \frac{\varepsilon}{2}$