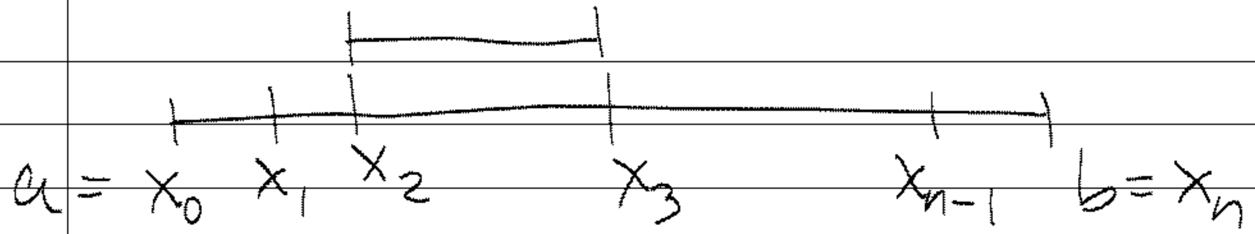
Riemman integration

Def: $a, b \in \mathbb{R}$ $a<b$. A partition of the closed interval $[a,b]$ is $a = x_0 < x_1 < \dots < x_n = b$



The width of the partition is $\max\limits_{1 \le i \le n} (x_i - x_{i-1})$

$f : [a,b] \longrightarrow \mathbb{R}$ A Riemman sum of f is

$$S = \sum_{i=1}^{n} f(\alpha_i)(x_i - x_{i-1}) \quad \text{where } \alpha_i \in [x_{i-1}, x_i]$$

Def: $f : [a,b] \longrightarrow \mathbb{R}$ f is Riemman integrable on $[a,b]$ if $\exists A \in \mathbb{R}$ such that, $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $|S - A| < \varepsilon$ $\forall$ Riemman sum S with width less than δ.

In this case A is called the Riemman integral of f between a and b and it is denoted by $A = \int_a^b f(x)\,dx$

Example: $f : [a,b] \longrightarrow \mathbb{R}$ $f(x) = k$. k a constant.

$a = x_0 < \cdots < x_n = b$ a partion $\alpha_i \in [x_{i-1}, x_i]$

$$\sum_{i=1}^{n} f(\alpha_i)(x_i - x_{i-1}) = k \sum_{i=1}^{n}(x_i - x_{i-1}) = k(b-a)$$

then $\int_a^b k\,dx = (b-a)k$

<u>Properties</u>: f & g real valued functions on $[a,b]$, $c \in \mathbb{R}$, f and g Riemman integrable. Then

1) $\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

2) $\int_a^b c f(x)\,dx = c \int_a^b f(x)\,dx$

<u>Proof</u>: 1) Let $\varepsilon > 0$. $\exists\ \delta_1 > 0$ & $\delta_2 > 0$ such that if S_1 and

S_2 are Riemman sums of f and g with width less than δ_1 & δ_2 respectively, then $\left|S_1 - \int_a^b f(x)dx\right| < \frac{\varepsilon}{2}$ and $\left|S_2 - \int_a^b g(x)dx\right| < \frac{\varepsilon}{2}$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Let $a = x_0 < \dots < x_n = b$ be a partition of width less than δ. Let $x_i \in [x_{i-1}, x_i]$.

$$\left|\sum_{i=1}^{n}\left[f(x_i) + g(x_i)\right](x_i - x_{i-1}) - \left(\int_a^b f(x)dx + \int_a^b g(x)dx\right)\right| \leq$$

$$\leq \left| \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) - \int_a^b f(x)\,dx \right| + \left| \sum_{i=1}^{n} g(x_i)(x_i - x_{i-1}) - \int_a^b g(x)\,dx \right| <$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$ because

1 is a Riemman sum of f with width $< \delta_1$ and

2 is a Riemman sum of g with width $< \delta_2$

Similar proof for 2)

Prop: $f : [a,b] \to \mathbb{R}$ integrable. $f(x) \geq 0 \quad \forall\, x \in [a,b]$.

then $\int_a^b f(x)dx \geq 0$.

pf: all the Riemman sums are non-negative. ~~then~~ Let $\varepsilon > 0$
Let S ~~bea~~ a Riemman sum such that $\left| S - \int_a^b f(x)dx \right| < \varepsilon$
then Δ ineq $\Rightarrow$ $\int_a^b f(x)dx > S - \varepsilon \geq -\varepsilon \quad \forall \varepsilon > 0$

$\Rightarrow \int_a^b f(x)\,dx \geq 0$.

Corollary: $f, g : [a,b] \to \mathbb{R}$ integrable.
$f(x) \geq g(x) \quad \forall \; x \in [a,b] \Rightarrow \int_a^b f(x)dx \geq \int_a^b g(x)dx$

pf: Apply last Prop to $f(x) - g(x)$

Corollary: $f : [a,b] \to \mathbb{R}$ integrable

$m \leq f(x) \leq M \quad \forall \; x \in [a,b] \implies$

$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

pf: Use last corollary with $f(x)$ and $g(x) = M$ and $f(x)$ and $g(x) = m$

Lemma: $f : [a,b] \to \mathbb{R}$. f integrable $\iff \forall \varepsilon > 0 \; \exists \; \delta > 0$ such that if S_1 and S_2 are two Riemman sums of f with width less than $\delta \Rightarrow |S_1 - S_2| < \varepsilon$

~~Pf~~: $\Rightarrow$) $\exists \delta > 0$: if S is RS with width $< \delta \Rightarrow$

$\left|S - \int_a^b f(x)\,dx\right| < \frac{\varepsilon}{2}$. Let S_1 & S_2 be two RS with width $< \delta$.

then $|S_1 - S_2| \leq \left|S_1 - \int_a^b f(x)\,dx\right| + \left|\int_a^b f(x)\,dx - S_2\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

($\Leftarrow$) Let $\beta_n > 0$ such that $|S^{(1)} - S^{(2)}| < \frac{1}{n}$ if $S^{(1)}$ & $S^{(2)}$ are RS with width $< \beta_n$. Let S_n be a RS with width $< \beta_n$.
and $\beta_n < \beta_m$ if $n > m$

Let $\varepsilon > 0$. Let $N > \frac{1}{\varepsilon}$. if $n, m \geq N$ $|S_n - S_m| < \frac{1}{N} < \varepsilon$

width $< \beta_n \le \beta_N$ width $< \beta_m \le \beta_N$

This shows S_n is a Cauchy sequence. $S_n \to S$

Let $\varepsilon > 0$. Let $N > \frac{1}{\varepsilon}$. If $\bar{S}$ is a RS with width $< \beta_N$ then

$$|\bar{S} - S| \le |\underbrace{\bar{S} - S_m}_{< \frac{1}{N} < \varepsilon}| + |\underbrace{S_m - S}_{\xrightarrow[m\to\infty]{} 0}| \le \varepsilon \quad \checkmark$$

$m \geq N$