Note Title 3/7/2016

Prop: $f: E \to \mathbb{R}^n$ $f(x) = (f_1(x), f_2(x), \dots, f_n(x))$

$f_i : E \to \mathbb{R}$. $x_0 \in E$. f is continuous at $x_0 \iff f_i$ is continuous at x_0 for all i

proof: $\Rightarrow$) Let $\varepsilon > 0$. Since f is continuous at x_0, $\exists\, \delta > 0$ such that $d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon$. But

$$d(f_i(x), f_i(x_0)) \leq d(f(x), f(x_0))$$

$$|f_i(x) - f_i(x_0)| \qquad\qquad \sqrt{\sum_{i=1}^{n} (f_i(x) - f_i(x_0))^2}$$

($\Leftarrow$) Let $\varepsilon > 0$. Since f_i is continuous at x_0, $\exists\ \delta_i > 0$ such that $d(x, x_0) < \delta_i \Rightarrow |f_i(x) - f_i(x_0)| < \frac{\varepsilon}{\sqrt{n}}$

Let $\delta = \min\limits_{1 \le i \le n} \delta_i$. If $d(x, x_0) < \delta \Rightarrow |f_i(x) - f_i(x_0)| < \frac{\varepsilon}{\sqrt{n}}$ for all i $\Rightarrow$ $d(f(x), f(x_0)) = \sqrt{\sum_{i=1}^{n} (f_i(x) - f_i(x_0))^2} < \sqrt{n \frac{\varepsilon^2}{n}} = \varepsilon$

Th: $f: E \to E'$ f is continuous. E compact. Then $f(E)$ is compact.

proof: Let U_i be open in $f(E)$ $\forall\ i \in I$ $(U_i \subset f(E))$

Assume $f(E) = \bigcup_{i \in I} U_i$. Let $V_i = f^{-1}(U_i)$. Since U_i is open and f is continuous then $V_i = f^{-1}(U_i)$ is also open

$$E = f^{-1}(f(E)) = f^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f^{-1}(U_i) = \bigcup_{i \in I} V_i$$

Since E is compact and each V_i is open, $\exists$ $i_1, \dots, i_n$ such that $E = V_{i_1} \cup V_{i_2} \cup \dots \cup V_{i_n}$ Apply f.

$$f(E) = f(V_{i_1}) \cup \dots \cup f(V_{i_n})$$

$V_i = f^{-1}(U_i) \Rightarrow f(V_i) = U_i$ (because f is onto $f(E)$, you check)

Then $f(E) = V_{i_1} \cup \ldots \cup V_{i_n} \Rightarrow f(E)$ is compact.

<u>Ex</u>: $E = \{ (a_n)_{n\in\mathbb{N}} :$ bounded sequences$\}$

$$d((a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}) = \sup_{n\in\mathbb{N}} |a_n - b_n|$$

d is a distance

$a_n = 0 \ \forall n \qquad C_1((a_n)_{n\in\mathbb{N}}) = \{ (b_n)_{n\in\mathbb{N}} : d((0)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}) \leq 1 \}$

is bounded, closed & complete but not compact.

A_1 1 0 0 0 0 0 0 ......

A_2 0 1 0 0 ....

A_3 0 0 1 0 ......

<u>Def</u>: $f: E \to \mathbb{R}$. We say that f attains its maximum at x_0 if $f(x_0) \geq f(x)$ $\forall$ $x \in E$. Similar for minimum

<u>Prop</u>: $f: E \to \mathbb{R}$, E compact, f continuous. f attains its maximum.

<u>proof</u>: E compact + f continuous $\Longrightarrow$ $f(E)$ compact = closed & bounded.

Let $a = \text{lub}(f(E))$, $a \in f(E)$ because $f(E)$ is closed.

$\Rightarrow \exists\, x_0 \in E$ such that $a = f(x_0)$ and since a is an ub of $f(E)$, we have $f(x_0) = a \geq f(x)$ $\forall\, x \in E$.

<u>Def</u>: f is uniformly continuous if $\forall \varepsilon > 0$ $\exists\, \delta > 0$ such that

$\forall x, y \in E$ such that $d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon$

<u>Ex</u>: 1) Uniformly continuous $f: E \to E$ $f(x) = x$

$\delta = \varepsilon$. If $d(x,y) < \delta \Rightarrow d(f(x), f(y)) = d(x,y) < \delta = \varepsilon$ ✓

2) Continuous but not uniformly continuous $f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$