Note Title 3/9/2016

Ex: $f(x)=x^2$ is continuous, but it is not uniformly continuous

proof: Assume it is uniformly continuous. Let $\varepsilon>0$. Assume $\delta>0$ is such that $|x-y|<\delta \Rightarrow |x^2-y^2|<\varepsilon$

$$y = x+\frac{\delta}{2} \qquad |y^2-x^2| = \left|\delta x+\frac{\delta^2}{4}\right| < \varepsilon$$

Assume $x>0$ $\qquad x+\frac{\delta}{4} < \frac{\varepsilon}{\delta} \qquad x < \frac{\varepsilon}{\delta} - \frac{\delta}{4}$

Th: $f: E \to E'$ continuous. E compact $\Rightarrow$ f is uniformly continuous

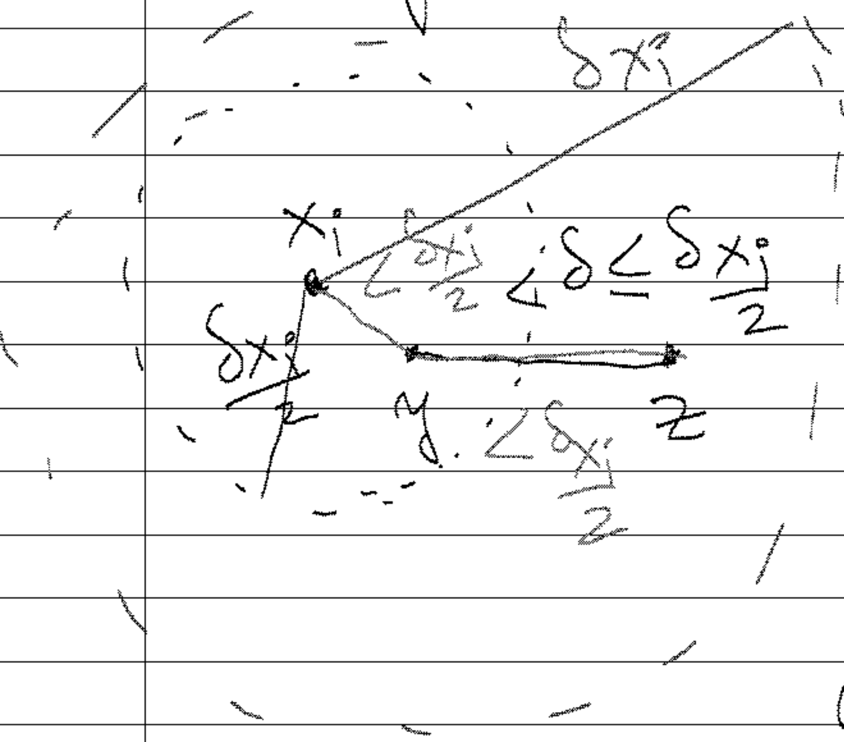
proof: Let $\varepsilon > 0$. $\forall x \; \exists \delta_x > 0$ such that $f(B_{\delta_x}(x)) \subset B_{\frac{\varepsilon}{2}}(f(x))$

$E = \bigcup_{x \in E} B_{\frac{\delta_x}{2}}(x)$. Since E is compact, $\exists \; x_1, x_2, \ldots, x_n$

such that $E = \bigcup_{i=1}^{n} B_{\frac{\delta_{x_i}}{2}}(x_i)$.

Let $\delta = \min_{1 \le i \le n} \frac{\delta_{x_i}}{2}$

Let $y, z \in E$ such that $d(y,z) < \delta$



(Let x_i such that $y \in B_{\frac{\delta_{x_i}}{2}}(x_i)$, we can do that because $E = \bigcup_{i=1}^{n} B_{\frac{\delta_{x_i}}{2}}(x_i)$)

$$d(z,y) < \delta \le \frac{\delta_{x_i}}{2}$$

$$d(y,x_i) \le \frac{\delta_{x_i}}{2}$$

$$d(x_i,z) < \delta_{x_i}$$

$$d(f(z),f(y)) \le d(f(z),f(x_i)) + d(f(x_i),f(y)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Ex: 1) $f(x) = \frac{1}{x}$ $\quad f: (0,1] \to \mathbb{R}$

is not uniformly continuous

2) $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0 \end{cases}$

is uniformly continuous because it is continuous and

$f: [0,1] \to \mathbb{R}$ and $[0,1]$ is compact.

Th: $f: E \to E'$, E connected $\Rightarrow f(E)$ connected
f continuous.

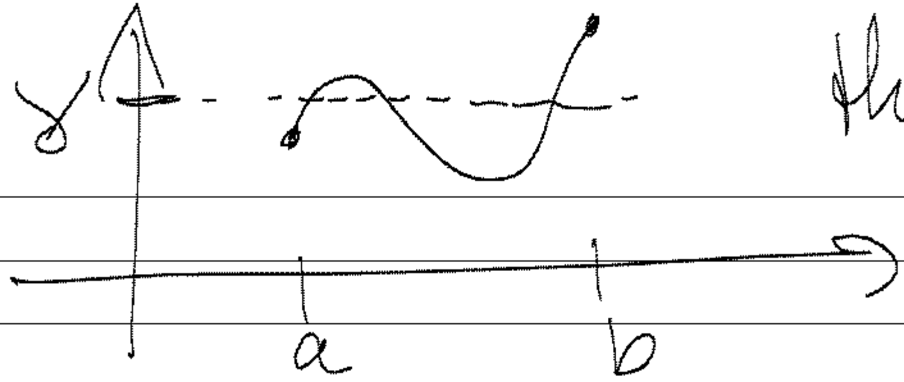
proof: Assume $f(E) = A \cup B$ both open in $f(E)$

$E = f^{-1}(A) \cup f^{-1}(B)$ $\quad f^{-1}(B)$ & $f^{-1}(A)$ are open because f is continuous

Since E is connected $\Rightarrow f^{-1}(A) = \phi$ or $f^{-1}(B) = \phi$ and thus,
$A = \phi$ or $B = \phi$

Corollary: $f: [a,b] \longrightarrow \mathbb{R}$ $\quad$ f continuous.

Let γ be between $f(a)$ and $f(b)$



then $\exists\, x \in (a,b)$ such $f(x) = \gamma$

<u>proof</u>: $f([a,b])$ is connected

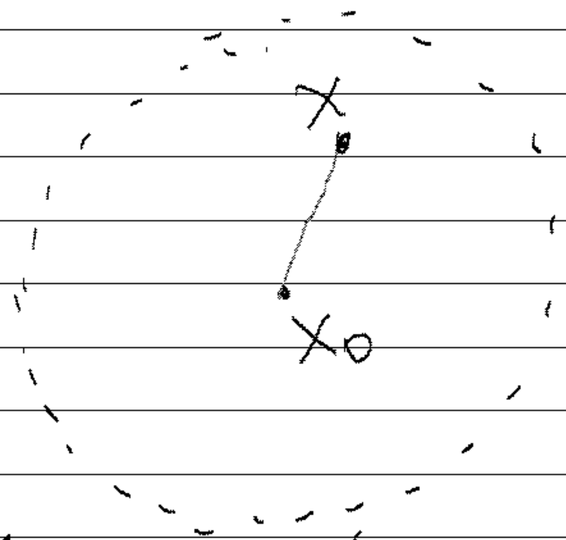
$f^{-1}((-\infty,\gamma)) \cup f^{-1}((\gamma,\infty)) \neq [a,b]$ because

$a \in$ open $\neq \emptyset$ $\quad$ $b \in$ open $\neq \emptyset$

Assume $f(a) < \gamma < f(b)$ | then $\exists\, c \in [a,b]$ such that $f(c) = \gamma$

<u>Obs</u>: Open balls in $\mathbb{R}^n$ connected

proof: Let $x_0 \in \mathbb{R}^n$ $r > 0$



Let $x \in B_r(x_0)$

$f_x : [0,1] \longrightarrow B_r(x_0)$

$f_x(t) = x_0 + t\,(x - x_0)$

Obs 1: f_x is continuous

Obs 2: $f_x([0,1]) \subset B_r(x_0)$ because $d\,(f_x(t), x_0) =$

$= \|f_x(t) - x_0\| = t\,\|x - x_0\| < t r \leq r$

Obs 3 $f_x([0,1])$ is connected

Obs 4 $f_x(0) = x_0$ thus $x_0 \in f_x([0,1]) \quad \forall x \in B_r(x_0)$

Obs 5 $f_x(1) = x$ thus $x \in f_x([0,1]) \quad \forall x$

then $B_r(x_0) = \bigcup_{x \in B_r(x_0)} f_x([0,1]) \implies$ connected (union of connected sharing a point)