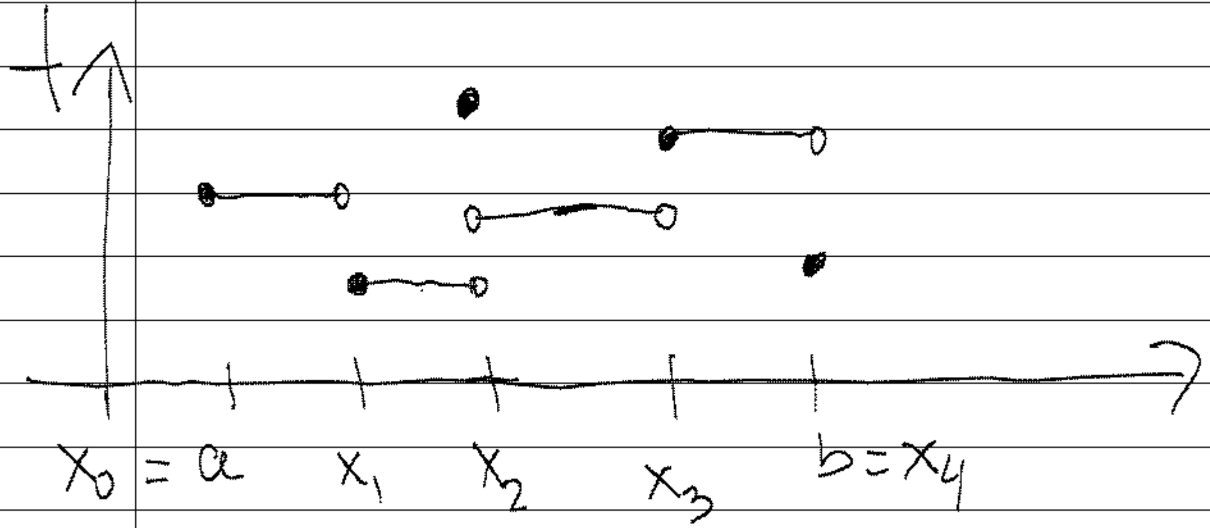
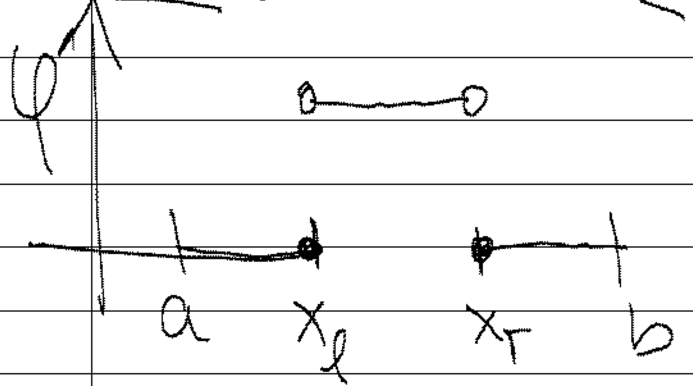
Note Title 4/8/2016

<u>Def</u>: A real valued function on the interval $[a,b]$ is called a step function if $\exists$ a partition $x_0 = a < x_1 < \cdots < x_n = b$ such that f is constant on each subinterval (x_{i-1}, x_i) $1 \le i \le n$



Obs: Let $a \le x_\ell < x_r \le b$. $\varphi : [a,b] \to \mathbb{R}$

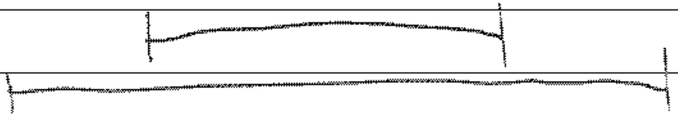


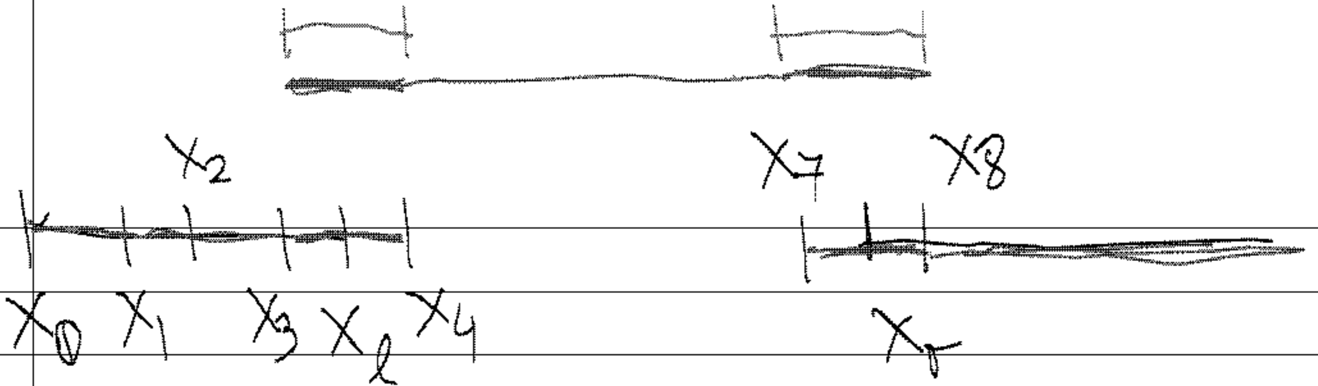
$$\varphi(x) = \begin{cases} 1 & \text{if } x \in (x_\ell, x_r) \\ 0 & \text{if } x \notin (x_\ell, x_r) \end{cases}$$

Claim: φ is integrable and $\int_a^b \varphi(x)\,dx = x_r - x_\ell$

proof: Let $\varepsilon > 0$. Let $\delta = \min\left\{ \frac{\varepsilon}{2}, \frac{x_r - x_\ell}{2} \right\}$

Let $a = x_0 < x_1 < \ldots < x_n = b$ be a partition with width $< \delta$





Let k_1 and k_2 such that $x_{k_1-1} < x_\ell < x_{k_1}$ and $x_{k_2-1} < x_r < x_{k_2}$

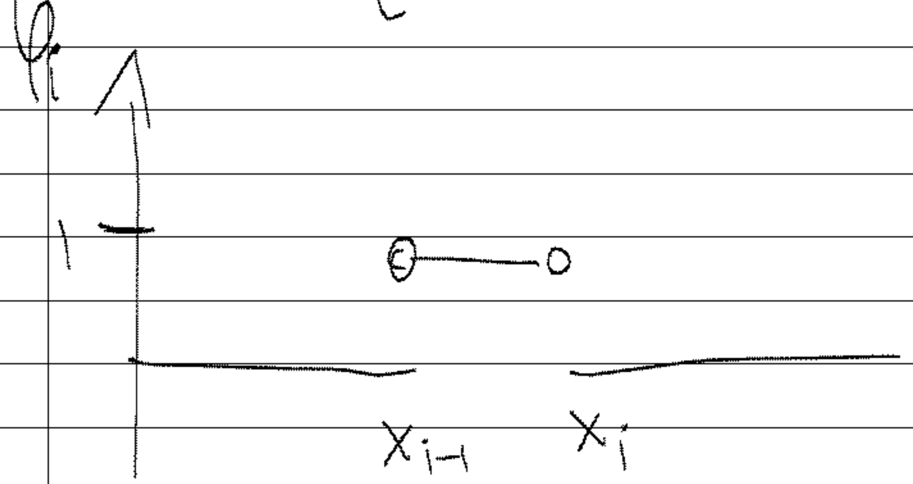
Then $\underbrace{x_{k_2-1} - x_{k_1}}_{> x_r - x_\ell - \varepsilon} \le S \le \underbrace{x_{k_2} - x_{k_1-1}}_{< x_r - x_\ell + \varepsilon}$ $\quad k_1 = 4 \quad k_2 = 8$

$$|S - (x_r - x_\ell)| < \varepsilon$$

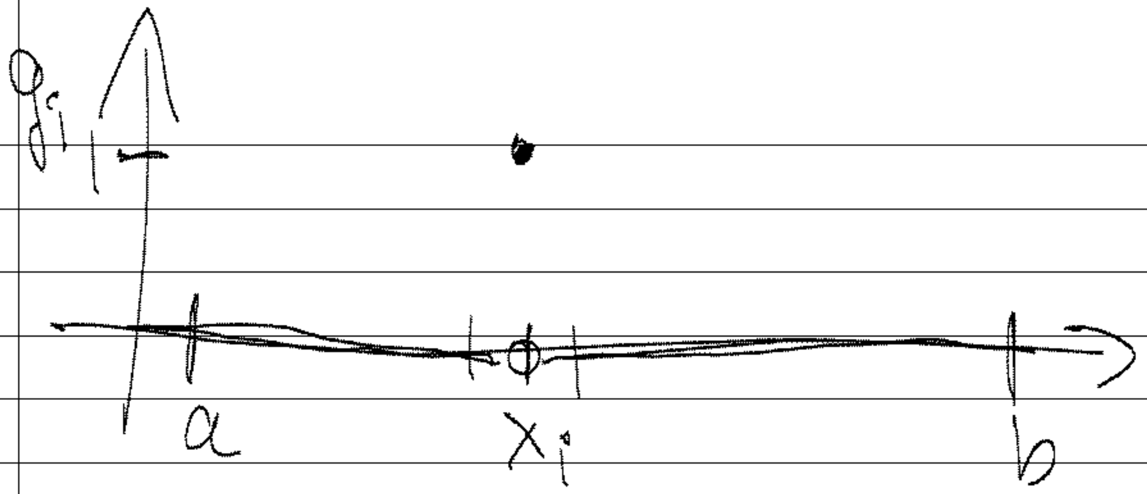
Obs: $f:[a,b]\longrightarrow\mathbb{R}$ a step fn. $a=x_0<\dots<x_n=b$

$$f(x)=\sum_{i=1}^{n} h_i\,\varphi_i(x)+\sum_{j=0}^{n}\beta_i\,g_i(x)$$

$$\varphi_i(x)=\begin{cases}1 & \text{if } x\in(x_{i-1},x_i)\\ 0 & \text{otherwise}\end{cases}\qquad g_i(x)=\begin{cases}1 & \text{if } x=x_i\\ 0 & \text{otherwise}\end{cases}$$



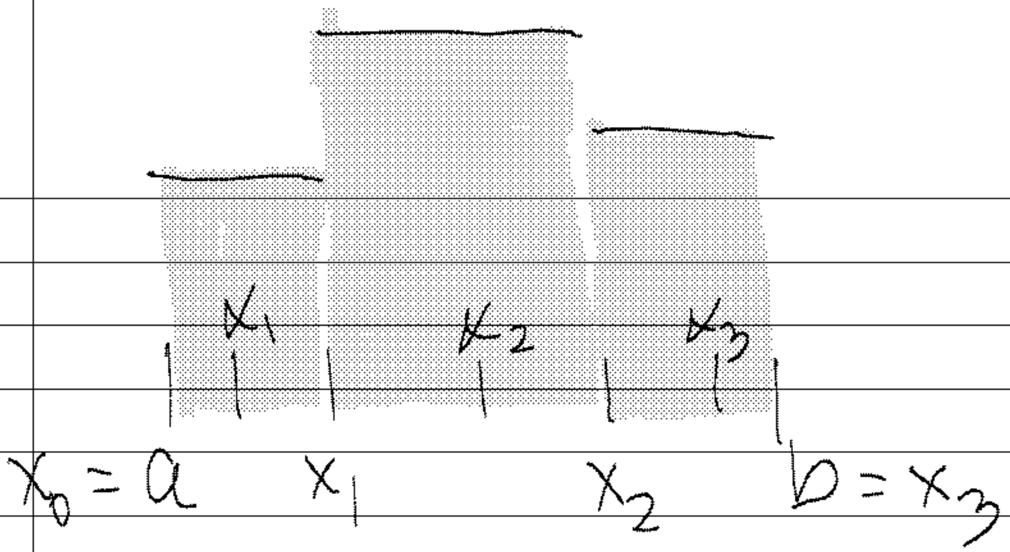
$$\boxed{\begin{aligned} h_i &= f\left(\frac{x_i+x_{i-1}}{2}\right)\\ \beta_i &= f(x_i)\end{aligned}}$$



$$\int_a^b g_i(x)\,dx = 0$$

f is integrable and $\displaystyle\int_a^b f(x)\,dx = \sum_{i=1}^{n} f(\xi_i)\,(x_i - x_{i-1})$

$$x_{i-1} < \xi_i < x_i$$



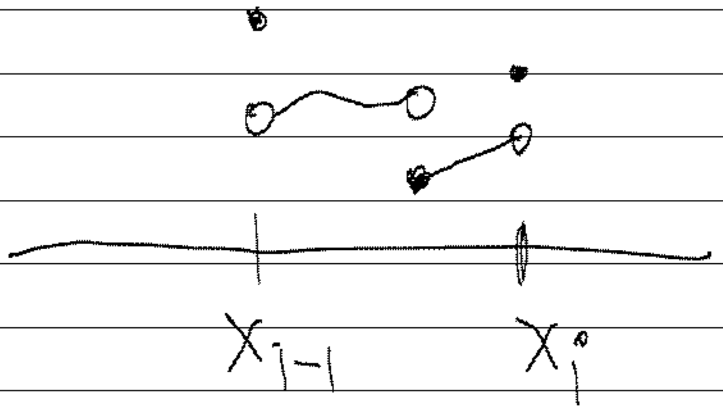
<u>Prop</u>: $f : [a,b] \longrightarrow \mathbb{R}$ $\quad$ f integrable on $[a,b] \iff \forall \varepsilon > 0$

$\exists$ f_1 & f_2 step fn such that $f_1(x) \le f(x) \le f_2(x)$

and $\int_a^b (f_2(x) - f(x))\, dx < \varepsilon$

proof $\Rightarrow$) f integrable. Let $\varepsilon > 0$. Let $\delta > 0$ such that if S is a Riemman sum of a partition with width less than δ, $\left| \int_a^b f(x)dx - S \right| < \beta$

Let $a = x_0 < \dots < x_n = b$ be a partition with width less than δ.



Let $\alpha_k \in [x_{k-1}, x_k]$ $\quad 1 \le k \le n$

i fixed

$$S = f(\alpha_i)(x_i - x_{i-1}) + \sum_{\substack{k=1 \\ k \neq i}}^{n} f(\alpha_k)(x_k - x_{k-1})$$

Since the width is less than δ, $\left|S - \int_a^b f(x)dx\right| < \varepsilon$

$$f(\alpha_i)(x_i - x_{i-1}) = S - \sum_{k\neq i} f(\alpha_k)(x_k - x_{k-1}) - \int_a^b f(x)dx + \int_a^b f(x)dx$$

$$|f(\alpha_i)|\,|x_i - x_{i-1}| < \varepsilon + \left|\sum_{k\neq i} f(\alpha_k)(x_k - x_{k-1})\right| + \left|\int_a^b f(x)dx\right|$$

α_k fixed $k \neq i$

this implies $f(\alpha_i)$ bounded in $[x_{k-1}, x_k]$

$M_i = \sup f(\alpha)$ $\qquad$ $m_i = \inf f(\alpha)$

$x \in [x_{i-1}, x_i]$ $\qquad$ $x \in [x_{i-1}, x_i]$

then $\quad m_i \le f(x) \le M_i \quad$ for $\quad x \in [x_{i-1}, x_i]$