Note Title 4/13/2016

Corollary: If $f:[a,b]\to\mathbb{R}$ is integrable, the f is bounded

Th: $f:[a,b]\longrightarrow\mathbb{R}$ continuous $\Longrightarrow$ f is integrable

proof: Let $\varepsilon>0$. Since $[a,b]$ is compact, f is uniformly continuous.
Then, $\exists\,\delta>0$: if $|x-y|<\delta$, $x,y\in[a,b]$ then $|f(x)-f(y)|<\dfrac{\varepsilon}{b-a}$

Let $a=x_0<x_1<\dots<x_n=b$ be a partition of width $<\delta$.

$m_i=\min\limits_{x\in[x_{i-1},x_i]} f(x)$ $\qquad M_i=\max\limits_{x\in[x_{i-1},x_i]} f(x)$ $\qquad m=\min\limits_{x\in[a,b]} f(x)$

$$M = \max_{x \in [a,b]} f(x) \qquad f_1(x) = \begin{cases} m_i & \text{if } x \in (x_{i-1}, x_i) \\ m & \text{if } x = x_0, x_1, \ldots, \text{or } x_n \end{cases}$$

$$f_2(x) = \begin{cases} M_i & \text{if } x \in (x_{i-1}, x_i) \\ M & \text{if } x = x_0, \ldots, \text{or } x_n \end{cases}$$

$$f_1(x) \leq f(x) \leq f_2(x) \qquad \forall\, x \in [a,b]$$

$$\int_a^b \left(f_2(x) - f_1(x) \right) dx = \sum_{i=1}^{n} \underbrace{(M_i - m_i)}_{< \frac{\varepsilon}{(b-a)}} (x_i - x_{i-1}) < \frac{\varepsilon}{\cancel{(b-a)}} \cancel{(b-a)} = \varepsilon$$

Prop: let $a<b<c$. $f:[a,c]\to\mathbb{R}$

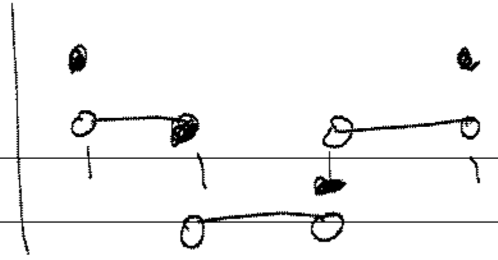
f is integrable on $[a,c] \iff f$ is integrable on $[a,b]$ and on $[b,c]$. In this case

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

proof: Case 1. $f(x)$ is a step function

$$a=x_0<x_1<\dots<x_{k-1}<b\le x_k<\dots<x_n=c$$

$$f(x)=\begin{cases} f_i & \text{if } \; x\in(x_{i-1},x_i) \\ h_i & \text{if } \; x=x_i \qquad 0\le i\le n \end{cases}$$



$$\int_a^c f(x)\,dx = \sum_{i=1}^{n} f_i\,(x_i-x_{i-1})$$

$$\int_a^b f(x)\,dx = \left[\sum_{i=1}^{k-1} f_i\,(x_i-x_{i-1})\right] + f_k(b-x_{k-1})$$

$$f:[a,b]\to\mathbb{R} \qquad f(x)=\begin{cases} f_i & \text{if } \; x\in(x_{i-1},x_i) \qquad i<k-1 \\ f_k & \text{if } \; x\in(x_{k-1},b) \\ h_i & \text{if } \; x=x_i \qquad 0\le i\le k-1 \\ \text{something at } b & \end{cases}$$

Similarly $\int_b^c f(x)\,dx = f_k(x_k - b) + \sum_{i=k+1}^{n} f_i(x_i - x_{i-1})$

Then $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

Case II. General f. $\Rightarrow$) Assume f is integrable on $[a,c]$.
We need to prove f is integrable on $[a,b]$ and on $[b,c]$
Let $\varepsilon > 0$. $\exists f_1$ & f_2 step functions on $[a,c]$ such that
$f_1(x) \le f(x) \le f_2(x)$ and $\int_a^c (f_2(x) - f_1(x))\,dx < \varepsilon$

Let $h_i(x) = f_i(x)$ $\quad x \in [a,b]$ $\quad h_i = f_i|$ $\quad i = 1,2$

$I[a,b]$

then $h_1(x) \leq f(x) \leq h_2(x) \quad x \in [a,b]$

Let $g_i = f_i \big|_{[b,c]} \qquad g_1(x) \leq f(x) \leq g_2(x) \qquad \forall\ x \in [b,c]$

$$\int_a^b (h_2(x) - h_1(x))\,dx + \int_b^c (g_2(x) - g_1(x))\,dx = \int_a^c (f_2(x) - f_1(x))\,dx < \varepsilon$$

then $\int_a^b (h_2(x) - h_1(x))\,dx < \varepsilon \quad \Rightarrow \quad \int_a^b f(x)\,dx$ & $\int_b^c f(x)\,dx$ exist.

$\int_b^c (g_2(x) - g_1(x))\,dx < \varepsilon$

$$\int_a^c f_1(x)\,dx < \int_a^c f(x)\,dx < \int_a^c f_2(x)\,dx < \int_a^c f_1(x)\,dx + \varepsilon$$

$$\int_a^c f_1 = \int_a^b f_1 + \int_b^c f_1 < \int_a^b f + \int_b^c f < \int_a^b f_2 + \int_b^c f_2 = \int_a^c f_2 < \int_a^c f_1 + \varepsilon$$

$\int_a^c f$ and $\int_a^b f + \int_b^c f$ are both in $\left(\int_a^c f_1, \int_a^c f_1 + \varepsilon\right)$

Since this is true $\forall \varepsilon > 0$, $\int_a^c f = \int_a^b f + \int_b^c f$

$\Leftarrow$) Let $\varepsilon > 0$. Let $h_i : [a,b] \to \mathbb{R}$ and $g_i : [b,c] \to \mathbb{R}$ step fns such that $h_1(x) \leq f(x) \leq h_2(x)$ $\forall x \in [a,b]$

$g_1(x) \le f(x) \le g_2(x) \quad \forall\, x \in [b,c]$ and $\int_a^b h_2 - h_1 < \frac{\varepsilon}{2}$

and $\int_b^c g_2 - g_1 < \frac{\varepsilon}{2}$

$$\beta_i(x) = \begin{cases} h_i(x) & x \in [a,b] \\ g_i(x) & x \in (b,c] \end{cases} \Rightarrow \beta_i \text{ are step fns such that } \int_a^c \beta_2 - \beta_1 < \varepsilon$$

and $\beta_1(x) \le f(x) \le \beta_2(x) \quad \forall\, x \in [a,c] \Rightarrow f$ is integrable on $[a,c]$

Again we can show that

$\int_a^c f$ and $\int_a^b f + \int_b^c f$ both are in $\left(\int_a^c \beta_1, \int_a^c \beta_1 + \varepsilon\right)$

$\forall \varepsilon > 0 \quad \Rightarrow \quad \int_a^c f = \int_a^b f + \int_b^c f$