Note Title 4/15/2016

Def: f integrable on $[a,b]$, we define

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

Corollary: $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

Note: If $|f(x)| \le M$ for all $x \in [a,b]$, then

$$\left|\int_a^b f(x)\,dx\right| \le M(b-a)$$

$U \subseteq \mathbb{R}$

<u>Theorem</u>: $f : U \to \mathbb{R}$. U open interval. f continuous, $a \in U$

Let $F(x) = \int_a^x f(t)\,dt$ $\forall$ $x \in U$. Then $F'(x) = f(x)$

<u>Pf</u>: $\left| \frac{F(x+h) - F(x)}{h} - f(x) \right| = \left| \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h} - f(x) \right| =$

$= \left| \frac{\int_x^{x+h} f(t)\,dt}{h} - f(x) \frac{\int_x^{x+h} dt}{h} \right| = \left| \frac{\int_x^{x+h} (f(t) - f(x))\,dt}{h} \right| \leq$

$$\leq \frac{\varepsilon(x+h-x)}{h} = \varepsilon \quad \text{we can do this } \forall \varepsilon > 0.$$

if $h < \delta$.

<u>Obs</u>: Any antiderivative of f is of the form

$$G(x) = \int_a^x f(t)\,dt + C$$

In other words: If $a \in U \subset \mathbb{R}$ U open interval.

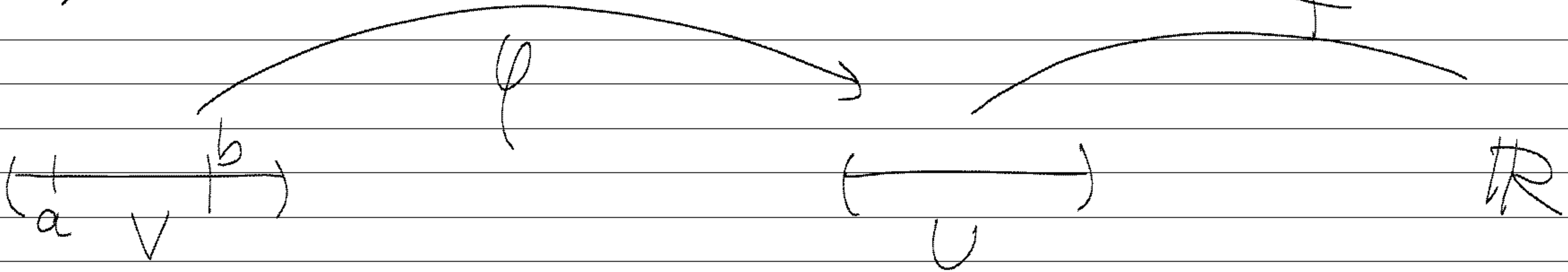
$f: U \to \mathbb{R}$ and $G: U \to \mathbb{R}$ and $f(x) = G'(x)$ then

$$\int_a^x f(t)\,dt = G(x) - G(a)$$

<u>Obs</u>: $f: U \to \mathbb{R}$ U open interval f continuous

$\varphi: V \longrightarrow U$ V open interval φ differentiable on V

$a, b \in V$



then $$\int_a^b f(\varphi(s))\,\varphi'(s)\,ds = \int_{\varphi(a)}^{\varphi(b)} f(t)\,dt$$

~~proof~~: $F(y) = \int_{\varphi(a)}^{y} f(t)\,dt$ $\qquad F'(y) = f(y)$

$G(x) = \int_a^x f(\varphi(s))\,\varphi'(s)\,ds$ $\qquad G'(x) = f(\varphi(x))\,\varphi'(x)$

$(F\circ\varphi)(x) = F(\varphi(x))$ $\qquad \frac{d}{dx}(F\circ\varphi)(x) = F'(\varphi(x))\,\varphi'(x) = f(\varphi(x))\,\varphi'(x)$

$$G'(x) = \frac{d}{dx}(F \circ \varphi)(x)$$

then

$$G(x) = F(\varphi(x)) + C \qquad \text{set } x = a \qquad G(a) = 0 \qquad F(\varphi(a)) = 0$$

then C

$$G(x) = F(\varphi(x)) \implies \int_a^x f(\varphi(s))\,\varphi'(s)\,dx = \int_{\varphi(a)}^{\varphi(x)} f(t)\,dt$$

for all x, then set $x = b$.

Def: If $x \in \mathbb{R}$, $x > 0$ $\log x = \int_1^x \frac{dt}{t}$

Prop: 1) $\frac{d}{dx} \log x = \frac{1}{x}$ $\forall\, x > 0$

2) $\log x$ is strictly increasing.

3) $\log(xy) = \log x + \log y$

proof: think of y as being a fixed number

$f(x) = \log(xy)$ $f'(x) = \frac{1}{xy} y = \frac{1}{x}$ $x > 0$

$g(x) = \log x + \log y$ $\qquad g'(x) = \frac{1}{x}$ $\qquad x > 0$

$f(x) = g(x) + C$ $\qquad$ set $x = 1$

$f(1) = \log y = g(1) = \log y \Rightarrow \quad C = 0$ ✓

4) $\log \frac{1}{x} = -\log x \qquad \forall \; x > 0$

$\log 1 = 0 = \log\left(\frac{1}{x} \cdot x\right) = \log \frac{1}{x} + \log x$ ✓

5) $\log x^n = n \log x \qquad n \in \mathbb{N} \qquad x > 0$

$\log x^n = \log(x\, x^{n-1}) = \log x + \log x^{n-1} = \log x + (n-1)\log x =$
$= n \log x.$

$n \geq 1.$ $n = 0$ $\log x^0 = 0 \log x = 0$ ✓

$n < 0$ $\log x^n = \log\left(\frac{1}{x}\right)^{-n} = -n \log \frac{1}{x} = -n(-1)\log x = n \log x$

$n \in \mathbb{Z}$

6) $\log x$ is surjective (onto) $\log 2 > 0$

$\log 2^n = n \log 2 \longrightarrow \infty$ as $n \to \infty$ $\Big\} \Longrightarrow$ $\log x$ is unbounded from

$\log\left(\frac{1}{2}\right)^n = -n\log 2 \longrightarrow -\infty$ as $n \to \infty$) above & below

$f(x) = \log x \qquad f : (0,\infty) \to \mathbb{R}$ one-one & onto.

then $\exists\ f^{-1} : \mathbb{R} \to (0,\infty)$

$f^{-1}(x) = \exp(x) = e^x$