Problem 1

We have 2 sets:

$$S_1 = \{a, x_1, x_2, ..., x_n, ...\}$$
$$S_2 = \{x_1, x_2, ..., x_n, ...\}$$

And we know the sequence x_i converges in E. Let's say it converges to the element x $\in E$.
We are going to show x = a, and that is because we are given S_1 is closed and S_2 is not closed nor open.

We have shown in class that if S_1 is a closed subset of metric space E, then any sequence in S_1 converges to an element in S. Assume $x \notin S_1$ and S_1 closed. This means $x \in S_1^C$ which is open so $\exists \epsilon : B_\epsilon(x) \subset S_1^C$. Which is a contradiction because for any ϵ we can find x_n in this ball and they are in S_1. Therefore we know:

$$\lim_{n \to +\infty} x_n = x \in S_1$$

Therefore as x is unique, it is either $x = x_i$ for some i. Or $x = a$.

We are going to show: $x = x_i$ is not possible.

Any sequence in S_2 is either finite or infinite. If finite, it converges to some x_i. If it is infinite, it has to be a subsequence of the sequence x_n.
We have shown in class that when a sequence x_n converges to x, then $\forall n_i$, the subsequence x_{n_i} also converges to x.
Therefore we know that if $\lim_{n\to\infty} x_n = x_i$, then all infinite sequences in S_2 converge to that x_i.
This would mean $\forall a_n \in S_2 lim a_n \in S_2$ which is $\leftrightarrow$ S_2 is closed. (Proven in class)

But we know S_2 is not closed, therefore we have a contradiction and we know $\lim_{n\to\infty} x_n \neq x_i$.

In conclusion:

$$\lim_{n \to +\infty} x_n = a$$

6

Problem 2

We have x_n be a sequence, we want to find a decreasing OR an increasing subsequence of x_n. There are two easy cases: (1) if x_n is unbounded from above, it has an increasing subsequence. (2) If it is unbounded from below, it has a decreasing subsequence. (3) If it is bounded it's a little trickier.

1. **$\mathbf{x_n}$ unbounded from above**:
   The definition for an unbounded sequence is that $\forall M, \exists k : x_k \geq M$
   So we start our increasing subsequence with: x_1.
   Now define: $M_1 = max\{x_1\}$. Then $\exists n_2 : x_{n_2} > M_1$. We know $n_2 > 1$, by definition of the fact it is strictly greater then the first element.
   From a first element we built the second element. Let's induct:
   Assume we have k elements that the beginning of an increasing sequence: $x_1 \leq x_{n_2} \leq x_{n_2} \leq ... \leq x_{n_k}$.
   Now take $M_k = max\{x_1, x_2, x_3, ..., x_{n_k}\}$. We know $\exists n_{k+1} : x_{n_{k+1}} > M_k$ (as x_n unbounded). We can also see that $n_{k+1} > n_k$. As we've chosen specifically an element strictly greater than all the previous ones. So now we have : $x_1 \leq x_{n_2} \leq x_{n_2} \leq ... \leq x_{n_k} \leq x_{n_{k+1}}$.
   We've shown by induction we can build an increasing subsequence.

2. **$\mathbf{x_n}$ unbounded from below**: This is similar to the previous argument except we can build a decreasing subsequence.
   Take x_1 as the first element. Then $\exists n_1 : x_{n_1} \leq x_1$, and $n_1 > 0$. Now we have a start of a decreasing subsequence: $\{x_1, x_{n_1}\}$.
   Assume we have k elements of a decreasing subsequence, such that: $x_1 \geq x_1 \geq x_2 \geq ... \geq x_{n_k}$.
   Now take: $L = min\{x_1, x_2, ..., x_{n_k}\}$. Then $\exists n_{k+1} : x_{n_{k+1}} < L$ (as x_n unbounded from below). Notice $n_{k+1} > n_k$ by definition as it can't be within the first n_k values.
   Now we have an extra element to add to our decreasing series.
   By induction, we can build a decreasing subsequence.

3. **$\mathbf{x_n}$ bounded**:
   We know that $\exists M, L : \forall n \in N : L \leq x_n \leq M$.

We want to show that either there is an increasing sequence in the set or a decreasing sequence.
Let's assume there is no increasing sequence, and show there is a decreasing one.
Lemma:
There is no increasing sequence $\rightarrow$ there is a maximal element in our set.
Proof:
Assume there is no maximal element in our set, let's build an increasing sequence.
Pick x_1 as the first element, just like in part 1, we know $\exists n_1 : x_{n_1} > x_1$, with $n_1 > 1$ (as otherwise, x_1 would be a maximal element).
Now we have: $\{x_1, x_{n_1}\}$ is the beginning of an increasing sequence.
Assume we have the first k elements to an increasing sequence: $\{x_1, x_{n_1}, x_{n_2}, ..., x_{n_k}\}$ Now take: $M = max\{x_1, x_{n_1}, x_{n_2}, ..., x$
We know M is some element in our set, therefore $\exists n_{k+1} > n_k : x_{n_{k+1}} > M$, as otherwise we would have a maximal element. Therefore have $\{x_1, x_{n_1}, x_{n_2}, ..., x_{n_k}, x_{n_{k+1}}\}$ is an increasing subsequence.
By induction, we have proved that there is an increasing sequence, which is a contradiction. This means there has to be a maximal element in our set.

Lemma:
A sequence has a maximal element $\rightarrow$ there exists a decreasing subsequence.
Proof:
We know our sequence has a maximal element: $\exists n_1 : x_{n_1} \geq x_n, \forall n$.
Now look at the sequence: $\{x_{n_1+1}, x_{n_1+2}, x_{n_1+3}, ...\}$.
Notice this new sequence is bounded and doesn't have an increasing sequence, therefore we know it has a maximal element again.
Now choose: $n_2 : x_{n_2} \geq x_n, \forall n > n_1$. We also know $x_{n_1} \geq x_{n_2}$ as x_{n_1} was maximal to all elements (therefore greater than or equal to x_{n_2}).

We can do this by induction: assume we have gotten the first k maximal elements: $x_{n_1} \geq x_{n_2} \geq x_{n_3} \geq ... \geq x_{n_k}$.
Now we look at the sequence: $\{x_{n_k+1}, x_{n_k+2}, x_{n_k+3}, ...\}$, Notice this new sequence is bounded and doesn't have an increasing sequence, therefore we know it has a maximal element again.
We therefore know: $\exists n_{k+1} : x_{n_{k+1}} \geq x_n, \forall n > n_k$. We also know: $x_{n_k} > x_{n_{k+1}}$, as x_{n_k} was maximal element to everything after index n_k.
So we added an element to our decreasing sequence.
This means we can build a decreasing sequence by induction.
Therefore we have shown that if the sequence is bounded and doesn't have an increasing subsequence, it has a decreasing subsequence.

Conclusion:
Our sequence always either has an increasing or decreasing subsequence.

Problem 3

What we know:

$$\forall x, \forall \epsilon > 0 : \exists \delta : d_1(x,y) < \delta \rightarrow d_2(x,y) < \epsilon$$

1. Here we are given $U \subset E$ is open in terms of d_2.
   This means, $\forall u \in U, \exists \epsilon > 0 : B2_\epsilon(u) \subset U$.
   Now we know that $\exists \delta > 0, d_1(u,y) < \delta \rightarrow d_2(u,y) < \epsilon$.
   This means, as all $y \in B2_\epsilon(u)$, and notice: $y \in B1_\delta(u)$.
   Therefore for any element in U, we have a ball of a certain radius (using d_1) contained in U. Therefore U is open with respect to d_1 too.
   Conclusion:
   A set $U \subset E$ open with respect to d_2, is also open with respect to d_1.

2. Let's explain what we have: $E = R, U = \{5\}$

$$d_1(x,y) = \begin{cases} 0, \text{if } x = y \\ 1, x \neq y \end{cases}$$

$$d_2(x,y) = |x - y|$$

Let's verify that the property holds:
$\forall \epsilon > 0, \forall x \in E$, we can pick a $\delta = 0.1$: $d_1(x,y) < \delta \rightarrow x = y \rightarrow d_2(x,y) = |x-y| = 0 < \epsilon$.

Nice. Now take the set: $U = \{3\}$. This set is open with distance d_1 as $B1_{0.5}(3) = \{3\} \subset U$. Which shows that you can put an open ball around the only element of the set, which means the set is open with respect to d_1. Now if you try to put a ball around 3 of positive radius, you will get elements out of our set, which means $\{3\}$ is not closed with respect to d_2.

3. We want to show:

$$\text{U connected with } d_1 \rightarrow \text{U connected with } d_2$$

By contrapositive, this is equivalent to showing:

$$\text{U } \textbf{not} \text{ connected with } d_2 \rightarrow \text{U } \textbf{not} \text{ connected with } d_1$$

Assume $U \subset E$ not connected with respect to d_2.
We have shown in class the Theorem:

$$\text{U not connected} \leftrightarrow U = A \cup B, A, B \text{ are open}, A \cap B = \varnothing$$

So we know $\exists A, B$ open sets with respect to d_2 that form $U = A \cup B$.
But we have proven in (1) that a set open with respect to d_2 is also open with respect to d_1.
Therefore sets A, B are open with respect to d_1.
And as $U = A \cup B, A \cap B = \varnothing$ still holds, with the previous theorem, we know U is not connected with respect to d_1.

4. The distances d_1, d_2 previously defined on R still work.
Take $U = (2, 3)$
We have shown in class in depth that an interval in R (with the standard d_2) is connected.
But take the set: $A = \{2.5\}, B = A^C \cap U = (2,3) - \{2.5\}$.
First notice: $A \cap B = \varnothing$. Also: $U = A \cup B$.
Let's show that those two sets are open with respect to d_1:

$$a \in A : B_{0.5}(a) = \{a\} \subset A$$

$$b \in B : B_{0.5}(b) = \{b\} \subset B$$

This means that both sets are open, and we have found 2 disjoint open sets that form U, which means U is not connected with respect to d_1, but it was connected with respect to d_2.

10

Problem 4

We have shown in Problem 2, a bounded sequence either has an increasing or decreasing subsequence.
It is understandable that this sequence has to converge due to the bounds.
We have proven in class the statement: "$a_n \in R$ decreasing and bounded then it converges". Let's repeat the proof.

1. **Assume $\mathbf{x_n}$ has a decreasing subsequence. Call it $\mathbf{x_{n_i}}$**
As x_n is bounded, we know the decreasing subsequence has a greatest lower bound: $g.l.b.\{x_{n_i}\} = g$.
Then we know $\forall \epsilon > 0 : g + \epsilon$ is not a lower bound (otherwise g is not a g.l.b).
This means: $\exists n_k : x_{n_k} \leq g + \epsilon$. And as x_{n_i} is decreasing, $\forall i > k : x_{n_i} \leq x_{n_k}$.
This means $\forall \epsilon > 0 : \exists k : \forall i \geq k, d(g, x_{n_i}) < \epsilon$.
Conclusion:

$$\lim_{k \to \infty} x_{n_k} = g$$

2. **Assume $\mathbf{x_n}$ has an increasing subsequence. Call it $\mathbf{x_{n_i}}$**
This is the same as the previous line, but we use $l = l.u.b\{x_{n_i}\}$.
For any $\epsilon > 0$: $\exists k : \forall i \geq k, d(x_{n_i}, l) < \epsilon$ (as it is increasing and bounded). Therefore:

$$\lim_{k \to \infty} x_{n_k} = l$$

Conclusion:
Any bounded sequence of real numbers has a converging subsequence.

6

Problem 5

We have proven in problem 4 that any bounded sequence has a converging subsequence.
Notice that when x_n is bounded, any subsequence is bounded. Therefore for any subsequence: x_{n_i}, there exists a subsubsequence that converges.
And the subsubsequence is a subsequence of x_n, therefore we know it has to converge to x^*.
This means for any subsequence, we can find something in there that converges to x^*.
We want to show:

$$\forall \epsilon > 0 : \exists N : n \geq N \rightarrow d(x^*, x_n) < \epsilon$$

Let's assume this is not true. Therefore: for some $\epsilon > 0$, we have an infinite number of points such that: $d(x^*, x_n) \geq \epsilon$.
All these points are a subsequence of x_n. Call this subsequence x_{n_i}. As it is bounded, we know it has a subsubsequence that converges.
Now because all: $\forall i, d(x_{n_i}, x^*) \geq \epsilon$, therefore our converging subsubsequence cannot converge to x^*, but this is a contradiction as we assumed all converging subsequences converge to x^*.
Conclusion:
If all converging subsequences of x_n converge to x^*, then x_n converges to x^*. ✓

Problem 6

I am going to use another example than the one given. It will use d_1 with $E = R$ from problem 3:

$$d_1(x,y) = \begin{cases} 0, \text{if } x = y \\ 1, x \neq y \end{cases}$$

We have shown earlier that any set with this metric is open. This is because:

$$B_{0.5}(x) = \{x\}, \forall x \in R$$

This means that any set's complement is also open, which means any set is closed too.
All sets are open and closed here.
Let's take the set: $S = (1,3)$, we know the set is both open and closed.
Notice as S is an open interval of R, it has an infinite number of elements.
It is bounded as: $\forall x \in S, 0 < x < 4$.
Now why is it not compact:

$$S = \bigcup_{x \in S} B_{0.5}(x)$$

Each open ball contains exactly one element: itself, and without that ball, we do not have that element. So we need each ball, and there is an infinite number of them.
Conclusion, the set $(1,3)$ is closed, bounded, but **not compact**.
That is in $E = R$ and with d_1 defined above.

✓

6

Total 40