Exam 1 - MATH 4317

The exam will be closed book, closed notes and no calculators will be allowed. Show all your work.

Which 3 problems do you want graded?

Last name:

First name:

Problem 1 (5 points): Let $E = \{x \in \mathbf{R} : x \geq 1\}$. For all $x, y \in E$, we define $d(x,y) = |1/x - 1/y|$.

1) Prove that d is a distance in E. (1 points)
2) Is the sequence $x_n = n$ Cauchy with this distance? (Prove your answer). (2 points)
3) Does the sequence $x_n = n$ converge? (Prove your answer). (2 points)

1) a) $d(x,y) \geq 0$ ✓ $d(x,y) = 0 \iff \left|\frac{1}{x} - \frac{1}{y}\right| = 0$

$\iff \frac{1}{x} - \frac{1}{y} = 0 \iff x = y$ ✓

b) $d(x,y) = d(y,x)$ ✓

c) $d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{1}{x} - \frac{1}{z} + \frac{1}{z} - \frac{1}{y}\right| \leq \left|\frac{1}{x} - \frac{1}{z}\right| + \left|\frac{1}{z} - \frac{1}{y}\right| =$

$= d(x,z) + d(z,y)$ ✓

2) ~~Yes. $|x_n - x_k| = \left|\frac{1}{n} - \frac{1}{k}\right|$. Le~~

$\boxed{\text{Yes.}}$ Let $\varepsilon > 0$, Let $N > \frac{2}{\varepsilon}$. If $n, k \geq N$

then $d(x_n, x_k) = \left|\frac{1}{n} - \frac{1}{k}\right| \leq \left|\frac{1}{n}\right| + \left|\frac{1}{k}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

3) $\boxed{\text{No}}$. proof by contradiction. ~~If~~ assume $x_n \to x$.

~~Let $\varepsilon =$~~ $x \in E$ thus $x \geq 1$. Let $\varepsilon = \frac{1}{2x}$

~~Let $\varepsilon > 0$.~~ No matter how large N is, let $n \geq N$

~~Let $N \geq$~~ such that $n > \frac{1}{\varepsilon}$, then $d(x_n, x) = \left|\frac{1}{x_n} - \frac{1}{x}\right|$

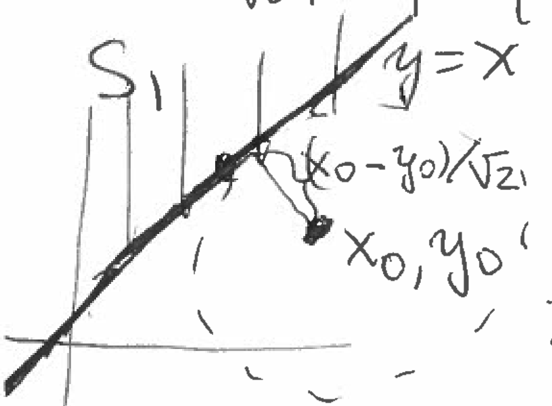
$$d(x_n, x) = \left|\frac{1}{x_n} - \frac{1}{x}\right| \geq \underbrace{\frac{1}{x}}_{= 2\varepsilon} - \frac{1}{x_n} = 2\varepsilon - \frac{1}{x_n} > \varepsilon$$

$$\frac{1}{x_n} < \varepsilon$$

thus $x_n \not\to x$

Problem 2 (5 points): Show that the subset of $\mathbf{R}^2$ given by $\{(x, y) : y \geq |x|\}$ is closed.

Let $S_1 = \{(x,y) : y \geq x\}$. We now prove that S_1 is closed. Let $(x_0, y_0) \notin S_1$ then $y_0 < x_0$. Let $\varepsilon = \frac{x_0 - y_0}{2}$.



If $(x,y) \in B_\varepsilon(x_0, y_0)$ then

$$d((x,y),(x_0,y_0)) = \sqrt{(x_0-x)^2 + (y_0-y)^2} < \varepsilon = \frac{x_0-y_0}{2}$$

then $|x - x_0| \leq \sqrt{(x_0-x)^2 + (y_0-y)} < \frac{x_0-y_0}{2}$

and $|y - y_0| < \frac{x_0-y_0}{2}$. then

$$y = y_0 + y - y_0 < y_0 + \frac{x_0-y_0}{2} = \frac{x_0+y_0}{2}$$

and $x = x_0 + x - x_0 > x_0 - \frac{(x_0-y_0)}{2} = \frac{x_0+y_0}{2}$ then

$x > y \Rightarrow (x,y) \notin S_1$. thus $B_\varepsilon(x_0,y_0) \subset S_1^c$

and thus S_1 is closed.

Let $S_2 = \{(x,y) : y \geq -x\}$. S_2 is closed (the proof is a with S_1)

$\Rightarrow S = S_1 \cap S_2$ is closed

Problem 3 (5 points): Prove that if $\lim_{n\to\infty} p_n = p$ in a given metric space then the set of points $\{p, p_1, p_2, p_3, \ldots\}$ is closed.

$S = \{p, p_1, p_2, p_3, \ldots\}$ ~~Let~~ Let $a \notin S$. Let $r = \frac{d(a,p)}{2}$.

~~Claim~~ Since $p_n \to p$ $\exists N$ such that $n \geq N$ then $d(p_n, p) < r$. Then, if $n \geq N$ $d(p_n, a) \geq$

$\geq d(a,p) - d(p_n, p) > 2r - r = r$. Thus

(reverse Δ inequality). $S \cap B_r(a) \subset \{p_1, \ldots, p_{N-1}\}$

~~Thus~~ Let $\varepsilon = \min\{r, d(a,p_1), \ldots, d(a, p_{N-1})\}$.

Then $S \cap B_\varepsilon(a) = \phi$. Thus, $B_\varepsilon(a) \subset S^c$ and

S is closed

Problem 4 (5 points): Show that if a sequence of numbers $a_1, a_2, a_3, \ldots$ converges to a, then

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n a_i}{n} = a$$

Let $\varepsilon > 0$. Let N such that $n \geq N \Rightarrow |a_n - a| < \frac{\varepsilon}{2}$

Let M $= \max\{|a_1 - a|, \ldots, |a_N - a|\}$. Then, if $n \geq N$

$$\left| \frac{\sum_{i=1}^n a_i}{n} - a \right| = \left| \frac{\sum_{i=1}^n (a_i - a)}{n} \right| \leq \frac{\sum_{i=1}^N |a_i - a|}{n} +$$

$$+ \frac{\sum_{i=N+1}^n |a_i - a|}{n} \leq \frac{NM}{n} + \frac{\varepsilon (n-N)}{2n} \leq$$

$\leq \frac{NM}{n} + \frac{\varepsilon}{2}$. Let $N_0 \geq \frac{2NM}{\varepsilon}$. Then,

if $n \geq N_0$, $\frac{NM}{n} \leq \frac{NM}{N_0} \leq \frac{\varepsilon}{2}$, which

implies that, if $n \geq N_0$

$$\left| \frac{\sum_{i=1}^n a_i}{n} - a \right| < \varepsilon .$$