Exam 2 - MATH 4317

The exam is to be done individually without the help of anyone. You are not to talk about the exam with anyone.

Last name:

First name:

Problem 1 (5 points): Let d and d' be two distances on the same set E so that the following is satisfied:

If $U \subset E$ is open with the distance d, then it is also open with the distance d'.

1) Let x_n be a sequence in E that converges with the distance d'. Prove that x_n also converges with the distance d.

2) Show an example of E, d, d' and x_n that satisfy the condition in italic letters, but x_n converges with the distance d but not with the distance d'.

Problem 2 (5 points): Let $f : \mathbb{R} \to \mathbb{R}$. Assume f is increasing, i.e. $x < y$ implies $f(x) < f(y)$. Let $x_0 \in \mathbb{R}$. Assume $f(x_0)$ is a cluster point of $f(\mathbb{R}) \cap [f(x_0), \infty)$ and $f(x_0)$ is also a cluster point of $f(\mathbb{R}) \cap (-\infty, f(x_0)]$. Prove that f is continuous at x_0.

Problem 3 (5 points): Let

$$S = \{0\} \cup \left\{ \frac{1}{n} : n \text{ positive integer} \right\}.$$

Is S compact or not? Prove it.

Problem 4 (5 points): Let E be a metric space. Let $S \subset E$. Assume S is compact. Let $f : E \to \mathbb{R}$ be defined as $f(x) = \text{g.l.b.}\{d(x, s) : s \in S\}$.

1) Prove that, for all $x \in E$ there exists $u \in S$ such that $f(x) = d(x, u)$.

2) Prove that f is continuous.