Problem 1) : 1) By assumption $\exists\, x \in E$ such that $x_n \to x$ with d'.

Claim 1) $x_n \to x$ also with d.

Let $\varepsilon > 0$. to proove claim 1) we need to show that $\exists N$ such that $n \geq N \Rightarrow$ $d(x_n, x) < \varepsilon$.

Let $B_\varepsilon(x) = \{y \in E : d(y, x) < \varepsilon\}$.

$B_\varepsilon(x)$ is open with d. Thus, by assumption, $B_\varepsilon(x)$ is also open with d'. Thus, since $x \in B_\varepsilon(x)$, $\exists\, \beta > 0$ such that

$$B'_\beta(x) = \{z \in E : d'(z, x) < \beta\} \subset B_\varepsilon(x)$$

Since $x_n \to x$ with d', $\exists N$ such that $n \geq N \Rightarrow x_n \in B'_\beta(x) \subset B_\varepsilon(x)$ and thus, $d(x_n, x) < \varepsilon \quad \forall n \geq N$. ✓

2) $E = \mathbb{R}$ $\quad x_n = \frac{1}{n}$

$d(x,y) = |x-y|$

$$d'(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

then $x_n \to 0$ in d,

but $d'(x_n, 0) = 1$ and thus, $x_n \not\to 0$ in d'

Also every set is open with d', thus, if U is open with d, U is open with d'

Problem 2: Let $\varepsilon > 0$ Since $f(x_0)$ is a cluster point of $f(\mathbb{R}) \cap [f(x_0), \infty)$, $\exists$ y_ε such that $y_\varepsilon \neq f(x_0)$ and

$$y_\varepsilon \in B_\varepsilon(f(x_0)) \cap \left(f(\mathbb{R}) \cap [f(x_0), \infty)\right)$$

this means $y_\varepsilon > f(x_0)$ (because $y_\varepsilon \neq f(x_0)$ and $y_\varepsilon \in [f(x_0), \infty)$). This also means $\exists$ x_ε such that $y_\varepsilon = f(x_\varepsilon)$. (This is because $y_\varepsilon \in f(\mathbb{R})$). ~~Since~~

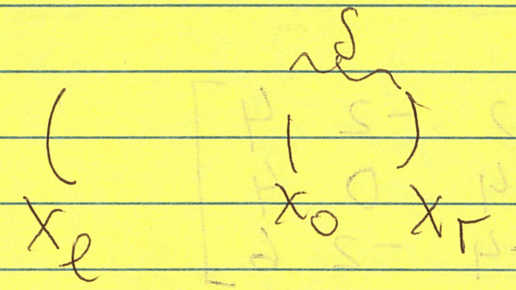
We also have $y_\varepsilon < f(x_0) + \varepsilon$ (because $y_\varepsilon \in B_\varepsilon(f(x_0))$. Since f is increasing and

$y_\varepsilon = f(x_\varepsilon) > f(x_0)$, we have $x_0 < x_\varepsilon$.

In summary, we have proved that $\forall$ $\varepsilon > 0$ $\exists$ $x_\varepsilon > x_0$ such that $f(x_0) < f(x_\varepsilon) < f(x_0) + \varepsilon$

Analogously, $\exists$ $x_\ell < x_0$ such that

$$f(x_0) - \varepsilon < f(x_\ell) < f(x_0).$$



Let $\delta = \min\{x_r - x_0, x_0 - x_\ell\}$

Then, if $|x - x_0| < \delta \Rightarrow x_\ell < x < x_r \Rightarrow$

$$f(x_0) - \varepsilon < f(x_\ell) < f(x) < f(x_r) < f(x_0) + \varepsilon$$

$\Rightarrow |f(x) - f(x_0)| < \varepsilon \Rightarrow f$ is continuous at x_0

Problem 3: $S^c = \bigcup_{n=1}^{\infty} \left(\frac{1}{n+1}, \frac{1}{n}\right) \cup$

$(-\infty, 0) \cup (1, \infty)$.

Thus S^c is the union of open intervals

Since open intervals are open sets & unions of open sets is open, S^c is open. Thus S is closed

$S \subset B_2(0)$. Thus S is a closed & bounded subset of $\mathbb{R}$. Thus S is compact

Problem 4:

1) Let $x \in E$. Let $g: S \to \mathbb{R}$ defined as $g(s) = d(s, x)$.

<u>Claim 1</u>: g is continuous.

<u>proof</u>: $|g(s_1) - g(s_2)| = |d(s_1, x) - d(s_2, x)|$

$\leq d(s_1, s_2)$

$\uparrow$

reverse triangle inequality.

Let $s_0 \in S$. Let $\varepsilon > 0$, select $\delta = \varepsilon$ to get

$d(s_0, s) < \delta \Rightarrow |g(s_0) - g(s)| < \varepsilon$. Thus

g is continuous at s_0. Since s_0 was an arbitrary point in S, g is continuous in S.

<u>Claim 2</u>: $\exists\, s_0 \in S$ such that $g(s_0) \leq g(s)$ $\forall s \in S$. ~~p~~

proof: Because g attains its minimum because g is continuous and S is compact (S is the domain of g).

We have proved that, $\forall x \in E$, $\exists s_0 \in S$ (s_0 depends on x) such that

$$g(s_0) = d(x, s_0) \le \underset{g(s)=}{d(x,s)} = \quad \forall s \in S$$

Thus, $\cancel{f_x}\ g(s_0) = d(x, s_0) = glb\{d(x,s) : s \in S\}$

i.e. $f(x) = d(x, s_0)$.

2) Let $x, y \in E$. Let $s_1, s_2 \in S$ such that $d(x, s_1) = f(x)$ & $d(y, s_2) = f(y)$

$$f(y) \le d(y, s_1) \le d(y,x) + d(x, s_1) = d(x,y) + f(x)$$

Thus $f(y) - f(x) \le d(x,y)$ analogously

$f(x) - f(y) \le d(x,y)$. Thus

$|f(x)-f(y)| \le d(x,y) \quad \forall x,y \in E$

Given $\varepsilon > 0$, select $\delta = \varepsilon$. to get

$d(x,y) < \delta \Rightarrow |f(x)-f(y)| < \varepsilon$ which

poovs f is continuous