Final - MATH 4317

Last name:

First name:

Problems you want graded: (Select exactly 4)

Each problem is worth 5 points

Problem 1: Prove that if $f : [a,b] \to \mathbb{R}$ is increasing then $\int_a^b f(x)dx$ exists.

Let N be a positive integer. Let $x_i = a + i\frac{(b-a)}{N}$ $0 \le i \le N$. Let

$$f_1(x) = \begin{cases} f(x_i) & \text{if } x \in [x_i, x_{i+1}) \quad 0 \le i \le N-1 \\ f(b) & \text{if } x = b \end{cases}$$

$$f_2(x) = \begin{cases} f(x_{i+1}) & \text{if } x \in (x_i, x_{i+1}] \quad 0 \le i \le N-1 \\ f(a) & \text{if } x = a \end{cases}$$

Claim 1: $f_1(x) \le f(x) \le f_2(x) \quad \forall x \in [a,b]$

Case 1, $x \in (x_i, x_{i+1}) \Rightarrow f_1(x) = f(x_i) \le f(x) \le$

$\le f(x_{i+1}) = f_2(x)$

~~Case 2~~ $\cancel{x = a \Rightarrow f_1(x) = f_2(x) = f(a)}$

Case 2 $\quad x = x_i \quad 0 \le i \le N \Rightarrow \cancel{f_1(x) = f(x_i)} =$

$f_1(x) = f(x) = f_2(x)$

$\cancel{= f(x) \le f(x_{i+1})}$

$$\xi_i = (x_i + x_{i-1})/2$$

$$\int_a^b (f_2(x) - f_1(x))dx = \sum_{i=1}^{N} (f_2(\xi_i) - f_1(\xi_i))(x_{i+1} - x_i)$$

$$= \sum_{i=1}^{N} (f(x_i) - f(x_{i-1}))\left(\frac{b-a}{N}\right) = \frac{(f(b) - f(a))(b-a)}{N}$$

$$< \varepsilon \quad \text{if} \quad f(b) = f(a) \quad \text{or}$$

$$N > \frac{(f(b) - f(a))(b-a)}{\varepsilon}$$

Problem 2: Prove that a differentiable real-valued function on $\mathbb{R}$ with bounded derivative is uniformly continuous.

Let $M > 0$ such that $M \geq |f'(x)|\ \forall x \in \mathbb{R}$.

$f(x) - f(y) = f'(z)(x-y)$ for some z between x and y $\Rightarrow$

$|f(x) - f(y)| = |f'(z)|\,|x-y| \leq M\ \cancel{(}|x-y|$

Let $\varepsilon > 0$. Select $\delta = \frac{\varepsilon}{M}$ $\Rightarrow$ $|x-y| < \delta$

$\Rightarrow |f(x) - f(y)| < \varepsilon$

Problem 3: Let $f, f_1, f_2, f_3, \cdots$ be continuous real-valued functions on a compact metric space E, with $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in E$. Prove that, if $f_1(x) \leq f_2(x) \leq f_3(x) \cdots$ for all $x \in E$, then the sequence $f_1, f_2, f_3, \cdots$ converges uniformly.

Let $\varepsilon > 0$. Let

$B_n = \{x \in E : f(x) - f_n(x) \geq \varepsilon\}$

Claim 1 B_n is closed $\forall n$

Claim 2 $B_{n+1} \subset B_n$

Claim 3 $\bigcap_{n=1}^{\infty} B_n = \emptyset$

Since E is compact, B_n closed, and $\bigcap_{n=1}^{\infty} B_n = \emptyset$, $\exists N$ such that $\bigcap_{n=1}^{N} B_n = \emptyset$ (from class). Since $B_N = \bigcap_{n=1}^{N} B_n$ we have $B_N = \emptyset$, which implies

$|f_k(x) - f(x)| < \varepsilon \quad \forall k \geq N \quad \forall x \in E$

thus $f_n \to f$ uniformly

Problem 4: Let E be metric space with more than one point. Let $x_0 \in E$. Assume $\{x_0\}$ is open. What can you say about the connectivity of E?

$\phi \neq \{x_0\} \neq E$ because E has more than one point.

$\{x_0\}$ is always closed. Since $\{x_0\}$ is also open in this case,

$$E = \underbrace{\{x_0\}}_{\text{open}} \cup \underbrace{E - \{x_0\}}_{\text{open}}$$

$\{x_0\} \cap (E - \{x_0\}) = \phi \Rightarrow E$ is not connected

Problem 5: Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is strictly increasing and onto. Prove that f and f^{-1} are continuous.

Let $x_0 \in \mathbb{R}$. Let $\varepsilon > 0$. Let x_l such that $f(x_l) = f(x_0) - \varepsilon$. Let x_r such that $f(x_r) = f(x_0) + \varepsilon$. x_l & x_r exist because f is onto. $x_l < x_0 < x_r$ because f is increasing. x_l & x_r are well defined because f is strictly increasing.

Let $\delta > 0$ such that $x_l \leq x_0 - \delta$ & $x_0 + \delta \leq x_r$. Then, if $|x - x_0| < \delta$, $\Rightarrow$ $x_l < x < \cancel{\ } x_r \Rightarrow f(x_l) < f(x) < f(x_r)$ $\Rightarrow |f(x) - f(x_0)| < \varepsilon \Rightarrow f$ is continuous

f strictly increasing & onto $\Rightarrow f^{-1}$ strictly increasing and onto. Thus, the conclusion above applies to f^{-1} and f^{-1} is also continuous

Problem 6: Show that if $f : E \to E'$ and $g : E' \to E''$ are both uniformly continuous, so is the composition $g \circ f : E \to E''$

Let $\varepsilon > 0$. Since g unif cont $\exists\, \beta > 0$ such that $d'(y_1, y_2) < \beta$ $\Rightarrow d''(g(y_1), g(y_2)) < \varepsilon$. Since f is uniformly continuous, $\exists\, \delta > 0$ such that $d(x_1, x_2) < \delta \Rightarrow d'(f(x_1), f(x_2)) < \beta$.

Thus, if $d(x_1, x_2) < \delta \Rightarrow d'(f(x_1), f(x_2)) < \beta \Rightarrow d''(g(f(x_1)), g(f(x_2))) < \varepsilon$. Thus $g \circ f$ is uniformly continuous