Math 4317 - Midterm Exam - Take home - Due on 2/23/15

Problem 1 (6 points): Let E be a metric space. let x_n be a convergent sequence in E and $a \in E$ such that the following are satisfied: 1) the set $\{a, x_1, x_2, \ldots, x_n, \ldots\}$ is closed, 2) the set $\{x_1, x_2, \ldots, x_n, \ldots\}$ is neither closed nor open. What can you say about $\lim_{n\to\infty} x_n$?. Prove any claim you make.

Problem 2 (6 points): Let x_n be a sequence of real numbers. Prove that there exists a subsequence x_{n_k} such that either $x_{n_{k+1}} \leq x_{n_k}$ for all k or $x_{n_{k+1}} \geq x_{n_k}$ for all k.

Problem 3 (10 points): Let E be a space and d_1 and d_2 two different metrics on E with the property that, for all $x \in E$ and all $\varepsilon > 0$, there exists $\delta > 0$ (that may depend on x), such that $d_1(x, y) < \delta$ impies that $d_2(x, y) < \varepsilon$. Let $U \subset E$. Prove the following:

1) If U is open with the distance d_2, then it is also open with the distance d_1

2) Give an example of distances d_1 and d_2 and $U \subset E$ such that U is open with the distance d_1 but not open with the distance d_2

3) Show that if U is connected with the distance d_1, then it is also connected with the distance d_2

4) Give an example of of distances d_1 and d_2 and $U \subset E$ such that U is connected with the distance d_2 but not connected with the distance d_1

Problem 4 (6 points): Let x_n be a bounded sequence of real numbers. Show that x_n has a convergent subsequence.

Problem 5 (6 points): Let x_n be a bounded sequence of real numbers. Show that, if all the convergent subsequence converges to the same limit $x^\star$, then x_n is convergent and converges to $x^\star$.

Problem 6 (6 points): Give an example of a metric space E, and $S \subset E$ such that S is bounded and closed but not compact. Prove all your claims. (Hint: you can take $E = \mathbb{R}^\infty = \{\mathbf{x} = (x_1, x_2, \ldots, x_n, \ldots) : x_i \in \mathbb{R}\}$ for each i and $x_n \neq 0$ only for a finite number of $n\}$.. Find the right distance and S. There are other options)