

MATH 6021 - Exercises/Homework

Problem 1: Show that $(f, g) = \int_0^1 f(x)g(x)dx$ defines an inner product in $C[0, 1]$.

Problem 2 (Due 5/27, Posted 5/16): Show that $\|x\|_1 = \sum_{i=1}^n |x_i|$ is a norm in \mathbb{R}^n .

Problem 3: Show that $\|f\|_1 = \int_0^1 |f(x)|dx$ is a norm in $C[0, 1]$. Is $\|f\|_\infty = \max_{0 \leq x \leq 1} |f(x)|$ also a norm in $C[0, 1]$?

Problem 4 (Due 5/27, Posted 5/16): Let (X, d) be a metric space. Let $\sigma(x, y) = d(x, y)/(1 + d(x, y))$. Prove that (X, σ) is a metric space.

Problem 5: Show that in a normed space $B_r(x) = x + rB_1(0)$.

Problem 6: Draw $B_\varepsilon(x)$, where the space is \mathbb{R}^2 and the distance is $d(x, y) = \|x - y\|_\infty = \max_{1 \leq i \leq 2} |x_i - y_i|$.

Problem 7: Prove that the open intervals in \mathbb{R} are open sets.

Problem 8: 1) Show that the open intervals in \mathbb{R} form a basis for the standard topology in \mathbb{R} . 2) Show that the open intervals with rational endpoints in \mathbb{R} also form a basis for the standard topology in \mathbb{R} .

Problem 9 (Due 5/27, Posted 5/18): Let (X, \mathcal{T}) be a topological space. Let \mathcal{B} be a basis for \mathcal{T} . Show that U is open if and only if U is the union of elements of \mathcal{B} .

Problem 10: Let (X, \mathcal{T}) be a topological space. Let $A \subseteq X$. Suppose that for each $x \in A$ there exists an open set U such that $x \in U \subseteq A$. Show that A is open.

Problem 11 (Due 5/27, Posted 5/20): Let X and Y be topological spaces. Let \mathcal{B} and \mathcal{C} be basis for the topology of X and Y respectively. Show that $\mathcal{D} = \{B \times C : B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$ is a basis for the topology of $X \times Y$.

Problem 12: Show that the standard topology in \mathbb{R}^2 is equal to the product topology when \mathbb{R}^2 is regarded as $\mathbb{R} \times \mathbb{R}$.

Problem 13: Let (X, \mathcal{T}) be a topological space and $Y \subseteq X$. Show that $\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$ is a topology on Y .

Problem 14: Let X be a topological space and Y a subspace of X . Show that, if U is open in Y and Y is open in X then U is open in X .

Problem 15: Is the following true or false? Let X be a topological space and Y a subspace of X . If U is open in Y then U is open in X . Provide a proof or a counterexample.

Problem 16: Let X be a topological space and Y a subspace of X . Show that if \mathcal{B} is a basis for the topology of X , then $\mathcal{B}_Y = \{Y \cap B : B \in \mathcal{B}\}$ is a basis for the topology on Y .

Problem 17: Let A be a subspace of X and B a subspace of Y . Show that the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

Problem 18: Show that if Y is a subspace of X , and A a subspace of Y , then the topology A inherits as a subspace of Y is the same as the topology A inherits as a subspace of X .

Problem 19: Show that $[a, b]$ is closed in \mathbb{R} and $\{(x, y) : x \geq 0 \text{ and } y \geq 0\}$ is closed in \mathbb{R}^2 .

Problem 20: Let $Y = [0, 1]$. Which of the following sets are open in Y ? Which are open in \mathbb{R} ?

$$\begin{aligned} A &= \{x \in \mathbb{R} : 1/2 < x \leq 1\}, \\ B &= \{x \in \mathbb{R} : 1/2 < x < 1\}, \\ C &= \{x \in \mathbb{R} : 1/2 \leq x < 1\}, \\ D &= \{x \in \mathbb{R} : 1/2 \leq x \leq 1\}. \end{aligned}$$

Problem 21: Let X be a topological space. Show that:

- 1) X and \emptyset are closed.
- 2) A_α closed for all $\alpha \in \Lambda$ implies that $\bigcap_{\alpha \in \Lambda} A_\alpha$ is closed.
- 3) A_1 and A_2 closed implies that $A_1 \cup A_2$ is closed.

Problem 22: Let X be a topological space and Y a subspace of X . Show that A is closed in Y if and only if A is the intersection of a closed set in X with Y .

Problem 23: Let X be a topological space and Y a subspace of X . Show that if A is closed in Y and Y is closed in X then A is closed in X .

Problem 24: Let X be a topological space and $A \subseteq X$. Show that: 1) G open and $G \subseteq A$ implies that $G \subseteq A^0$ and 2) F closed and $A \subseteq F$ implies that $\overline{A} \subseteq F$.

Problem 25: Let X be a topological space and $A \subseteq X$. Show that: 1) A open implies that $A^0 = A$ and 2) A closed implies that $\overline{A} = A$.

Problem 26: Let $(X, \|\cdot\|)$ be a normed space, $x \in X$ and $r > 0$. Show that $\overline{B_r(x)} = \{y \in X : \|x - y\| \leq r\}$.

Problem 27: Let X be a topological space and Y a subspace of X and $A \subseteq Y$. Let \overline{A} be the closure of A in X . Show that the closure of A in Y is $\overline{A} \cap Y$.

Problem 28: Let X be a topological space and \mathcal{B} a basis. Let $A \subseteq X$ and $x \in X$. Show that $x \in \overline{A}$ if and only if every $B \in \mathcal{B}$ that contains x intersects A .

Problem 29 (Due 6/6, Posted 5/23): Let X be a topological space. Let B , A and A_α ($\alpha \in \Lambda$) be subsets of X . Show that:

- 1) $B \subseteq A$ implies $\overline{B} \subseteq \overline{A}$.
- 2) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
- 3) $\overline{\bigcup_{\alpha \in \Lambda} A_\alpha} \subseteq \bigcup_{\alpha \in \Lambda} \overline{A_\alpha}$.

Problem 30: What is the limit set and closure of $\{1/n : n \text{ positive integer}\}$?

Problem 31: Let (X, d) be a metric space. Let $x_n \in X$ for all positive integer n . Show that $x_n \rightarrow x$ if and only if for all $\varepsilon > 0$ there exists a positive integer N such that $n \geq N$ implies $d(x_n, x) < \varepsilon$.

Problem 32 (Due 6/6, Posted 5/23): Let X be a Hausdorff topological space. Let A be subsets of X . Show that $x \in A'$ if and only if every neighborhood of x contains an infinite number of points of A .

Problem 33: Prove that the product of two Hausdorff topological spaces is Hausdorff and a subspace of a Hausdorff space is Hausdorff.

Problem 34: Prove that the norms $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$, $\|x\|_1 = \sum_{i=1}^n |x_i|$ and $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$ induce the same topology in \mathbb{R}^n .

Problem 35 (Due 6/6, Posted 5/25): Let X be a normed space, $x \in X$ and $r > 0$. Show that $bd(B_r(x)) = \{y \in X : \|y - x\| = r\}$.

Problem 36: Show that $A^0 \cap bd(A) = \emptyset$ and $\overline{A} = A^0 \cup bd(A)$.

Problem 37: Show that U is open if and only if $bd(U) = \overline{U} - U$.

Problem 38: Find the boundary and the interior of the following sets

$$A = \{(x, y) : y = 0\}$$

$$B = \{(x, y) : \}$$

$$C = A \cup B$$

$$D = \{(x, y) : x \text{ is irrational}\}$$

$$E = \{(x, y) : 0 < x^2 - y^2 \leq 1\}$$

$$F = \{(x, y) : x \neq 0 \text{ and } y \leq 1/x\}$$

Problem 39: Let X, Y be a topological spaces. Prove that $f : X \rightarrow Y$ is continuous if and only if $f^{-1}(V)$ is open for all $V \subseteq Y$ open if and only if $f^{-1}(F)$ is closed for all $F \subseteq Y$ closed.

Problem :40 Let X, Y be a topological spaces. Let $f : X \rightarrow Y$ and $x \in X$. Let \mathcal{B}_x and \mathcal{B}_y be neighborhood basis at x and $f(x)$ for the topology of X and Y respectively. Prove that $f : X \rightarrow Y$ is continuous at x if and only if for every $B_y \in \mathcal{B}_y$ there exists $B_x \in \mathcal{B}_x$ such that $f(B_x) \subseteq B_y$.

Problem 41: Let X, Y be a topological spaces. Let \mathcal{B} be basis for the topology of Y . Prove that $f : X \rightarrow Y$ is continuous if and only if $f^{-1}(B)$ is open for all $B \in \mathcal{B}$.

Problem 42: Show that $f : \mathbb{R} \rightarrow \mathbb{R}^2$ given by $f(x) = (3x^2, 2x)$ is continuous.

Problem 43: Let X, Y be a topological spaces. Let $f : X \rightarrow Y$ be a bijection. Show that the following are equivalent:

- 1) f is a homeomorphism

- 2) $f(U)$ is open in Y if and only if U is open in X .
 3) V is open in Y if and only if $f^{-1}(V)$ is open in X .

Problem 44: Let X, Y, Z be topological spaces. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be continuous. Show that $g \circ f : X \rightarrow Z$ (i.e. $g \circ f(x) = g(f(x))$) is continuous.

Problem 45: Let X, Y, Z be topological spaces. Show that if X is homeomorphic to Y and Y is homeomorphic to Z , then X is homeomorphic to Z .

Problem 46: Let $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \text{ and } (x, y) \neq (1, 0)\}$. Show that A is homeomorphic to \mathbb{R} and \mathbb{R} is homeomorphic to $(0, 1)$.

Problem 47: Let X, Y be topological spaces. Let $f : X \rightarrow Y$ be continuous and bijective. Show that $f(G)$ is open for every open G implies that f is a homeomorphism.

Problem 48: Let X, Y be topological spaces. Let $f : X \rightarrow Y$ be continuous and bijective. Does $f(F)$ is closed for every closed F imply that f is a homeomorphism? (Justify).

Problem 49 (Due 6/6, Posted 5/25): Let (X, d) be a metric space. Show that $d : X \times X \rightarrow \mathbb{R}$ is continuous ($X \times X$ with the product topology).

Problem 50 (Due 6/10, Posted 5/25): Let (X, d) be a metric space. Let $A \subseteq X$ prove that the function $f(x) = \text{dist}(x, A) = \inf_{a \in A} d(x, a)$ is continuous.

Problem 51 (Due 6/10, Posted 5/25): Let (X, d) be a metric space. Let $A \subseteq X$. Show that $x \in \overline{A}$ if and only if there exists a sequence $\{a_n\} \subseteq A$ such that $a_n \rightarrow x$.

Problem 52: Let (X, d) be a metric space. Let $A \subseteq X$. Let $d' = d|_{A \times A}$, the restriction of d to pairs of elements in A . Show that (A, d') is a metric space and its topology is the same as the topology that A inherits from X as a subspace of X .

Problem 53: Let (X, d_X) and (Y, d_Y) be metric spaces. Let $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Let $\rho((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$. Show that ρ is a distance for $X \times Y$ and that the topology induced by this distance is equal to the product topology.

Problem 54 (Due 6/10, Posted 5/27): Let X be a topological space and (Y, d) be metric space. Let $f_n : X \rightarrow Y$ be a sequence of continuous functions and x_n a sequence of points in X . Show that $x_n \rightarrow x$ and f_n converges uniformly to f implies that $f_n(x_n) \rightarrow f(x)$.

Problem 55: Let X, Y be homeomorphic topological spaces. Show that X is connected if and only if Y is connected.

Problem 56: Let X, Y be connected topological spaces. Show that $X \times Y$ is connected. (Hint: Let $(a, b) \in X \times Y$. Let $T_y = X \times \{y\} \cup \{a\} \times Y$. Show that T_y is connected for all $y \in Y$. Show that $(a, b) \in T_y$ for all $y \in Y$. Show that $X \times Y = \cup_{y \in Y} T_y$)

Problem 57 (Due 6/10, Posted 5/27): Let X be a topological space. Let $A_n \subseteq X$

(n positive integer) A_n connected for all n . Assume that $A_n \cap A_{n+1} \neq \emptyset$ for all n . Show that $\cup_{n=1}^{\infty} A_n$ is connected.

Problem 58: A space is totally disconnected if its only connected subspaces are one-point sets. Show that if X has the discrete topology, then X is totally disconnected. Does the converse hold? (Justify).

Problem 59: Let X be a topological space. Let $A \subseteq X$. Show that if C is a connected subspace of X that intersects both A and $X - A$, then C intersects $bd(A)$.

Problem 60: Let X, Y be connected topological spaces. Let $A \subseteq X, B \subseteq Y$, with $A \neq X$ and $B \neq Y$. Show that $X \times Y - A \times B$ is connected.

Problem 61: Let X be a connected topological space. Let $f : X \rightarrow \mathbb{R}$ continuous. Let $a, b \in X$ such that $f(a) \leq f(b)$. Show that for all $r \in [f(a), f(b)]$ there exists $x \in X$ such that $f(x) = r$.

Problem 62 (Due 6/17, Posted 6/1): Let X be a topological space. Show that X path connected implies X connected.

Problem 63: Show that:

- a) $[0, 1]$ is path connected.
- b) $\mathbb{R}^2 - \{(0, 0)\}$ is path connected.
- c) Let X be a normed space, $x \in X$ and $r > 0$. $B_r(x)$ is path connected.
- d) $\{(x, \sin(1/x) : x > 0\} \cup \{(0, 0)\} \times [-1, 1]$ is connected but not path connected.

Problem 64: Let X, Y be topological spaces and $f : X \rightarrow Y$ an homeomorphism. Show that, for any $A \subseteq X$, $X - A$ is homeomorphic to $Y - f(A)$.

Problem 65 (Due 6/17, Posted 6/1): Show that no two of the spaces $(0, 1)$, $(0, 1]$ and $[0, 1]$ are homeomorphic.

Problem 66: Show that \mathbb{R}^n and \mathbb{R} are not homeomorphic if $n > 1$.

Problem 67 (Due 6/17, Posted 6/1): Let $S^1 = \{x \in \mathbb{R}^2 : \|x\|_2 = 1\}$. Let $f : S^1 \rightarrow \mathbb{R}$ be continuous. Show that there exists $x \in S^1$ such that $f(x) = f(-x)$.

Problem 68: Let $f : [0, 1] \rightarrow [0, 1]$ be continuous. Show that there exists $x \in [0, 1]$ such that $f(x) = x$ (such x is called a fixed point of f).

Problem 69 (Due 6/17, Posted 6/1): True or false (justify):

- a) X, Y path connected implies $X \times Y$ path connected
- b) $A \subseteq X$. A path connected implies \overline{A} path connected
- c) X path connected and $f : X \rightarrow Y$ implies $f(X)$ path connected
- d) $A_\alpha \subseteq X$ path connected for all $\alpha \in \Lambda$ and $\cap_{\alpha \in \Lambda} A_\alpha \neq \emptyset$ implies that $\cup_{\alpha \in \Lambda} A_\alpha$ is path connected

Problem 70 (Due 6/24, Posted 6/1): Let $X = A \cup B$, where A and B are closed in

X . Let $f : A \rightarrow Y$ and $g : B \rightarrow Y$ be continuous functions such that $f(x) = g(x)$ for all $x \in A \cap B$. Let

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B. \end{cases}$$

Show that h is well defined and h is continuous.

Problem 71: Let X be topological space and $x, y, z \in X$. Show that if there is a path in X from x to y and a path in X from y to z then there is path in X from x to z .

Problem 72 (Due 6/24, Posted 6/8): Let Y be a compact subspace of a Hausdorff space X and x_0 not in Y . Show that there exists U and V open sets of X such that $Y \subseteq U$, $x_0 \in V$ and $V \cap U = \emptyset$.

Problem 73: Let X and Y be homeomorphic topological spaces. Show that X is compact if and only if Y is compact.

Problem 74 (Due 6/24, Posted 6/10): Let X be a topological space. Show that X is compact if and only if for every collection of closed sets $\{C_\alpha\}_{\alpha \in \Lambda}$ that has the finite intersection property we have $\bigcap_{\alpha \in \Lambda} C_\alpha \neq \emptyset$.

Problem 75 (Due 6/24, Posted 6/10): Let \mathcal{T} and \mathcal{T}' be two topologies for X . Assume that X is compact and Hausdorff for both topologies and that $\mathcal{T} \subseteq \mathcal{T}'$, i.e. $U \in \mathcal{T}$ implies $U \in \mathcal{T}'$. Show that $\mathcal{T} = \mathcal{T}'$.

Problem 76: Let X be a topological space. Show that a finite union of compact subspaces of X is compact.

Problem 77: Let (X, d) be a metric space and Y a subspace of X . Show that if Y is compact, then Y is closed and bounded.

Problem 78 (Due 7/1, Posted 6/10): Let (X, d) be a metric space and Y a subspace of X . Does Y being closed and bounded imply that Y is compact? (Justify, i.e. proof or counterexample).

Problem 79 (Due 7/1, Posted 6/10): Let A and B be disjoint compact subspaces of a Hausdorff space X . Show that there exists disjoint open sets (in X) U and V such that $A \subseteq U$ and $B \subseteq V$.

Problem 80: Let X, Y be topological spaces, with X compact and Y Hausdorff, and $f : X \rightarrow Y$ continuous. Let $A \subseteq X$. Show that A closed implies that $f(A)$ is closed.

Problem 81: Let X, Y be topological spaces, with Y compact Hausdorff, and $f : X \rightarrow Y$. Show that f is continuous if and only if $G_f = \{(x, f(x)) : x \in X\}$ (the graph of f) is closed in $X \times Y$.

Problem 82 (Due 7/1, Posted 6/10): Let X be a topological space. Let $f : X \rightarrow \mathbb{R}$ and $g : X \rightarrow \mathbb{R}$ be two continuous functions. Show that $h(x) = \max\{f(x), g(x)\}$ is continuous.

Problem 83 (Due 7/1, Posted 6/13): Let (X, d) be a metric space. Let $A \subseteq X$ ($A \neq \emptyset$) and $x \in X$. Show that:

- a) $\text{dist}(x, A) = 0$ if and only if $x \in \overline{A}$
- b) A compact implies there exists $a \in A$ such that $d(x, a) = \text{dist}(x, A)$
- c) A compact, $U \subseteq X$, U open and $A \subseteq U$ implies there exists $\varepsilon > 0$ such that $\{x \in X : \text{dist}(x, A) < \varepsilon\} \subseteq U$.
- d) c) may not hold if A is closed but not compact

Problem 84 (Due 7/11, Posted 6/13):

- a) Show that $f : (0, 1) \rightarrow \mathbb{R}$ given by $f(x) = 1/x$ is continuous but not uniformly continuous.
- b) Let $f : (0, 1) \rightarrow \mathbb{R}$ be given by $f(x) = \sqrt{x}$. Is f continuous? Is f uniformly continuous?. (justify, i.e. proof or counter example).

Problem 85: Let (X, d) be a metric space. Let $f : X \rightarrow X$ be an isometry, i.e. $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. Show that X being compact implies that f is bijective.

Problem 86 (Due 7/11, Posted 6/15): Let (X, d) be a metric space. Let $f : X \rightarrow X$ be a contraction, i.e. there exists $r < 1$ such that $d(f(x), f(y)) \leq rd(x, y)$ for all $x, y \in X$. Show that X being compact implies that f has unique fixed point, i.e. there exist a unique $x \in X$ such that $f(x) = x$.

Problem 87 (Due 7/11, Posted 6/20): Show that every Cauchy sequence is bounded.

Problem 88: Let X be a topological space and (Y, d) a bounded metric space. Show that $(CB(X, Y), \rho)$ is a metric space where, as defined in class, $CB(X, Y) = \{f : X \rightarrow Y : f \text{ is continuous and bounded (i.e. } f(X) \text{ is bounded)}\}$ and $\rho(f, g) = \sup_{x \in X} d(f(x), g(x))$ for all $f, g \in CB(X, Y)$.

Problem 89 (Due 7/11, Posted 6/20): Let X be a metric space.

- a) Assume there exists $\varepsilon > 0$ such that $\overline{B_\varepsilon(x)}$ is compact for all $x \in X$. Show that X is complete.
- b) Assume that for every $x \in X$ there exists $\varepsilon > 0$ such that $\overline{B_\varepsilon(x)}$ is compact. Show by means of an example that X need not be complete.

Problem 90: Let (X, d_X) and (Y, d_Y) be metric spaces. Let Y be complete. Let $A \subseteq X$. Show that if $f : A \rightarrow Y$ is uniformly continuous, then f can be uniquely extended to a continuous function $g : \overline{A} \rightarrow Y$, and g is uniformly continuous.

Problem 91: Let (X, d) be a metric space. Show that X is complete if and only if for all collection of closed sets $\{A_n\}_{n \in \mathbb{N}}$ such that $A_{n+1} \subseteq A_n$ and $\text{diam}(A_n) \rightarrow 0$ we have $\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset$.

Problem 92: Let (X, d) be a metric space. A map $f : X \rightarrow X$ is called a contraction if there exists $r < 1$ such that $d(f(x), f(y)) \leq rd(x, y)$ for all $x, y \in X$. Show that if X is complete and f a contraction, there exists $x \in X$ such that $f(x) = x$.

Problem 93: Prove that the vector space $CB([0, 1], \mathbb{R})$ is not finite dimensional.

Problem 94: Let (X, d) be a metric space. Let $a, x_0 \in X$. Let $f : X \rightarrow \mathbb{R}$ given by $f(x) = d(x, a) - d(x, x_0)$ is bounded and continuous.

Problem 95: Let X and Y be two normed spaces. Show that $(BL(X, Y), \|\cdot\|)$ is a normed space, where, as defined in class, $\|T\| = \inf_{\{M \geq 0: \|T(x)\| \leq M\|x\| \text{ for all } x \in X\}} M$.

Problem 95: Let X and Y be two normed spaces. Let $T \in BL(X, Y)$. Show that

$$\|T\| = \sup_{x \neq 0} \frac{\|T(x)\|}{\|x\|} = \sup_{\|x\|=1} \|T(x)\|.$$

Problem 96 (Due 7/18, Posted 6/22): Let $X = \mathcal{C}[0, 1] = \{f : [0, 1] \rightarrow \mathbb{R} \text{ continuous}\}$ with the norm $\|f\| = \max_{x \in [0, 1]} |f(x)|$. Let $T : X \rightarrow X$ be defined as $T(f)(x) = \int_0^x f(t) dt$. Show that $T \in BL(X, X)$ and compute $\|T\|$.

Problem 97: Let $A \in \mathbb{R}^{m \times n}$. Show that $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by $T(x) = Ax$ is a bounded linear operator.

Problem 98 (Due 7/18, Posted 6/22): Let X be a vector space, (\cdot, \cdot) an inner product in X and $\|\cdot\|$ the induced norm. Let $a \in X$. Let $T : X \rightarrow \mathbb{R}$ defined by $T(x) = (a, x)$. Show that $T \in X^*$ and $\|T\| = \|a\|$.

Problem 99 (Due 7/18, Posted 6/27): Let X be a real normed space. Let $f : X \rightarrow \mathbb{R}$ be a linear functional, $f \neq 0$. Let $\text{Ker}(f) = f^{-1}(\{0\})$. Show the following:

- 1) f is bounded if and only if $\text{Ker}(f)$ is closed
- 2) f is not bounded if and only if $\text{Ker}(f)$ is dense.

Problem 100: Let v_1, \dots, v_n be a basis of a finite dimensional vector space X . For each $x \in X$, let $\lambda_i(x)$ be the unique numbers that satisfy $\sum_{i=1}^n \lambda_i(x)v_i = x$. Let $\|x\|_1 = \sum_{i=1}^n |\lambda_i(x)|$ and $S = \{x \in X : \|x\|_1 = 1\}$. Show that $\|\cdot\|_1$ is a norm and S is compact (in the topology induced by this norm).

Problem 101 (Due 7/18, Posted 6/27): Let $\|\cdot\|$ and $\|\cdot\|'$ be two norms on the same space X . Show that these two norms are equivalent if and only if there exists positive numbers m and M such that $m\|x\| \leq \|x\|' \leq M\|x\|$ for all $x \in X$.

Problem 102: Let X and Y be two normed spaces. Let $T : X \rightarrow Y$ be a bounded linear operator. Show that if (x_n) is a Cauchy sequence in X , then $(T(x_n))$ is a Cauchy sequence in Y .

Problem 103: Let (X, d) be a metric space. Let $F \subseteq X$ closed and $K \subseteq X$ compact such that $F \cap K = \emptyset$. Show that $\text{dist}(K, F) = \inf_{x \in K, y \in F} d(x, y) > 0$.

Problem 104: Let (X, d) be a metric space. Does $F_1, F_2 \subseteq X$ closed and $F_1 \cap F_2 = \emptyset$ imply $\text{dist}(F_1, F_2) > 0$? (Justify).

Problem 105: Let X be a metric space. Let $\varepsilon > 0$ and (x_n) be a sequence in X that satisfies $d(x_n, x_m) \geq \varepsilon$ whenever $n \neq m$. Show that (x_n) does not have any convergent subsequence.

Problem 106: Let $A \subseteq \mathbb{R}^n$. Show that A is a flat if and only if $\lambda a + \mu b \in A$ whenever $a, b \in A$ and $\lambda + \mu = 1$.

Problem 107: Let $A_\alpha \subseteq \mathbb{R}^n$ be a flat for all $\alpha \in \Lambda$. Show $\bigcap_{\alpha \in \Lambda} A_\alpha$ is a flat.

Problem 108: Let $A \subseteq \mathbb{R}^n$. Show that

$$\text{aff}(A) = \bigcap_{F \text{ flat}: F \supseteq A} F$$

Problem 109: Let $A \subseteq \mathbb{R}^n$ be a flat. Let $a_1, \dots, a_m \in A$ and $\lambda_1, \dots, \lambda_m$ such that $\sum_{i=1}^m \lambda_i = 1$. Show that $\sum_{i=1}^m \lambda_i a_i \in A$.

Problem 110: Let $A \subseteq \mathbb{R}^n$ be a flat. Assume $A = a_1 + S_1 = a_2 + S_2$ where $a_1, a_2 \in \mathbb{R}^n$ and S_1 and S_2 are subspaces of \mathbb{R}^n . Show that $S_1 = S_2 = A - a$, where a is any element of A .

Problem 111: Let $A \subseteq \mathbb{R}^n$. Show that $\text{aff}(A)$ is the set of all affine combinations of points of A .

Problem 112: Show that a_0, a_1, \dots, a_r are affinely independent if and only if $a_1 - a_0, \dots, a_r - a_0$ are linearly independent.

Problem 113: Show that a_0, a_1, \dots, a_r are affinely independent if and only if $\dim(\text{aff}(\{a_0, a_1, \dots, a_r\})) = r$.

Problem 114: Let a_0, a_1, \dots, a_r be affinely independent. Let $x \in \text{aff}(\{a_0, a_1, \dots, a_r\})$. Show that there exists unique numbers $\lambda_0, \lambda_1, \dots, \lambda_r$ such that $x = \sum_{i=0}^r \lambda_i a_i$ and $\sum_{i=0}^r \lambda_i = 1$.

Problem 115: Show that $H \subseteq \mathbb{R}^n$ is a hyperplane if and only if there exists $a \in \mathbb{R}^n$ and $c \in \mathbb{R}$ with $a \neq 0$ such that $H = \{x \in \mathbb{R}^n : (a, x) = c\}$.

Problem 116: Show that flats are convex sets.

Problem 117: Let $A \subseteq \mathbb{R}^n$ be a flat. Let $H \subseteq \mathbb{R}^n$ be a hyperplane that intersects A but does not contain A . Show that $A \cap H$ is a flat and that $\dim(A \cap H) = \dim(A) - 1$.

Problem 118: Let $a \in \mathbb{R}^n$ and $c \in \mathbb{R}$ with $a \neq 0$. Let $R = \{x \in \mathbb{R}^n : (a, x) \geq c\}$. Show that R is convex.

Problem 119: Show that the intersection of convex sets is convex.

Problem 120: Let A be convex. Show that the convex combination of points in A is in A .

Problem 121: Let $A \subseteq \mathbb{R}^n$. Show that

$$\text{conv}(A) = \bigcap_{B \text{ convex: } B \supseteq A} B.$$

Problem 122: Let $A \subseteq \mathbb{R}^n$. Show that $\text{conv}(A)$ is the set of all convex combination of points in A .

Problem 123: What the the convex hull of the following sets:

- a) $\{x \in \mathbb{R}^n : \|x\|_2 = 1\}$
- b) $\{a, b\}$ where $a, b \in \mathbb{R}^n$
- c) $\{(0, 0), (0, 1), (1, 0), (1/3, 1/3)\}$

Problem 124: Which of the following sets is convex:

- a) $\{(x, y) : y \geq 2x^2 \text{ and } y \leq x^2 + 1\}$
- b) $\{(x, y) : y \geq 2x^2 \text{ and } y \leq -x^2 + 1\}$

Problem 125 (Due 7/25, Posted 7/1): Show that in \mathbb{R}^n , the convex hull of a compact set is compact and the convex hull of an open set is open.

Problem 126: Let $A \subseteq X$ convex. Let $x, y \in X - A$. Prove that $x \in \text{conv}(A \cup \{y\})$ and $y \in \text{conv}(A \cup \{x\})$ implies $x = y$.

Problem 127: Let a_0, a_1, \dots, a_r be affinely independent. Show that $\text{ri}(\text{conv}(\{a_0, a_1, \dots, a_r\})) = \{\sum_{i=0}^r \lambda_i a_i : \sum_{i=0}^r \lambda_i = 1 \text{ and } \lambda_i > 0 \text{ for all } i\}$.

Problem 128: Let $A, B \subseteq \mathbb{R}^n$ convex sets. Assume that $A \subseteq B$ and $\dim(A) = \dim(B)$. Show that $\text{ri}(A) \subseteq \text{ri}(B)$.

Problem 129 (Due 7/25, Posted 7/1): Let $A \subseteq \mathbb{R}^n$ convex. Let $a, b \in A$ and $r > 0$. Show that $B_r(a) \cap \text{aff}(A) \subseteq A$ implies that $B_{\lambda r}(\lambda a + (1 - \lambda)b) \cap \text{aff}(A) \subseteq A$ for all $\lambda \in (0, 1]$.

Problem 130: Let $A \subseteq \mathbb{R}^n$ convex. Let $a \in \text{ri}(A)$ and $b \in \overline{A}$. Show that $\lambda a + (1 - \lambda)b \in \text{ri}(A)$ for all $\lambda \in (0, 1]$.

Problem 132: Let $A \subseteq \mathbb{R}^n$ convex. Show that $\text{ri}(A)$ is convex and \overline{A} is convex.

Problem 133: Let $A \subseteq \mathbb{R}^n$ closed and $x \in \mathbb{R}^n$. Show that there exists $a_0 \in A$ such that $\|x - a_0\| = \text{dist}(x, A)$.

Problem 134: Let $A, B \subseteq \mathbb{R}^n$ be convex sets. Show that $A - B = \{a - b : a \in A \text{ and } b \in B\}$ is convex.

Problem 135: Let $A \subseteq \mathbb{R}^n$ be convex. Let $x \in \text{rb}(A)$. Show that there exists a hyperplane H that supports A at x and $A \not\subseteq H$.

Problem 136: Let $Q = [-1, 1]^2 = \{(x, y) : -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1\}$. Let f be the projection operator of Q . What if $f(x, y)$ (i.e. give the formula for f)?

Problem 137: Let $A, B \subseteq \mathbb{R}^n$ be disjoint convex sets. Let $x \in \mathbb{R}^n$. Show that $\text{conv}(A \cup \{x\}) \cap B = \emptyset$ or $\text{conv}(B \cup \{x\}) \cap A = \emptyset$.

Problem 138: Show that every compact convex set in \mathbb{R}^n can be expressed as an intersection of closed balls.

Problem 139: Let

$$A = \{(x, y) : y \geq 2x^2 \text{ and } y \leq x^2 + 1\} \text{ and } B = \{(0, 3/2)\}.$$

Prove that A and B can not be separated by a hyperplane.

Problem 140 (Due 7/25, Posted 7/8): Let $A \subseteq \mathbb{R}^n$ be closed and convex. Let f be the projection operator of A . Let $x, y \in \mathbb{R}^n$ and $a \in A$. Assume that $f(x) = f(y) = a$ prove that $f(\lambda x + (1 - \lambda)y) = a$ for all $0 \leq \lambda \leq 1$.

Problem 141: Let $A \subseteq \mathbb{R}^n$ be convex. Show the following:

- 1) The intersection of a family of faces of A is a face of A .
- 2) If B is a face of A and C is a face of B then C is a face of A .
- 3) If H is a hyperplane that supports A then $H \cap A$ is a face.

Problem 142: Let $A \subseteq \mathbb{R}^n$ be convex and F a face of A . Show that $x \in F$ and $x = \sum_{i=1}^r \lambda_i a_i$ with $\sum_{i=1}^r \lambda_i = 1$ and $\lambda_i > 0$ and $a_i \in A$ for all i implies that $a_i \in F$ for all i .

Problem 143 (Due 7/25, Posted 7/11): Which closed convex sets have no extreme points in \mathbb{R}^2 ?

Problem 144: Let V be the set of vertices of a polytope P in \mathbb{R}^n and let x be a point not lying in $\text{aff}(P)$. Show that $\text{conv}(\{x\} \cup P)$ is a polytope whose set of vertices is $\{x\} \cup V$.

Problem 145: Let a_0, \dots, a_r be affinely independent. Show that F is a face of $\text{conv}(\{a_0, \dots, a_r\})$ if and only if $F = \text{conv}(\{a_i : i \in I\})$ for some $I \subseteq \{0, \dots, r\}$.

Problem 146: Let a_0, \dots, a_r be affinely independent. Let $I \subseteq \{0, \dots, r\}$ and $J \subseteq \{0, \dots, r\}$. Show that $I = J$ if and only if $\text{conv}(\{a_i : i \in I\}) = \text{conv}(\{a_i : i \in J\})$.

Problem 147: Let $B = \{x \in \mathbb{R}^2 : \|x\|_2 < 1\}$. True or false: For every $f : B \rightarrow B$ continuous there exists $x \in B$ such that $f(x) = x$. (Justify).

Problem 148: Let $K \subseteq \mathbb{R}^n$ compact and convex. Let $d = \dim(K)$. Show that K is homeomorphic to $\{x \in \mathbb{R}^d : \|x\|_2 \leq 1\}$.

Problem 149: Show that $\{x \in \mathbb{R}^n : \|x\|_2 < 1\}$ is homeomorphic to \mathbb{R}^n .