MATH 6021
HOMEWORK SET 5

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Problem 70. Let $X = A \cup B$, where A and B are closed in X. Let $f: X \to Y$ and $g: B \to Y$ be continuous functions such that $f(x) = g(x)$ for all $x \in A \cap B$. Let

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B. \end{cases}$$

Show that h is well-defined and h is continuous.

Proof. We see that h is well-defined since for any point $x \in A \cap B$, we have $f(x) = g(x)$. To see that h is continuous, let C be a closed set in Y. Then

$$h^{-1}(C) = \{x \in X \mid h(x) \in C\} = \{a \in A \mid f(a) \in C\} \cup \{b \in B \mid g(b) \in C\} = f^{-1}(C) \cup g^{-1}(C).$$

Since f is continuous on A, $f^{-1}(C)$ is closed in A, and thus is closed in X since A is closed in X. Similarly, $g^{-1}(C)$ is closed in X. Thus $h^{-1}(C)$ is closed in X as a finite union of closed sets. □

Problem 72. Let Y be a compact subspace of a Hausdorff space X and x_0 not in Y. Show that there exists U and V open sets of X such that $Y \subseteq U$, $x_0 \in V$ and $V \cap U = \{\}$.

Proof. For each $y \in Y$, there is a neighborhood U_y of y and neighborhood V_y of x_0 such that $U_y \cap V_y = \{\}$. Then $\{U_y\}_{y \in Y}$ is an open cover of Y, and since Y is compact, there is a finite subcollection $\{U_i\}_{i=1}^n$ that covers Y. If we take the neighborhoods of x_0 corresponding to the U_i and call them V_i, then we have that $U := \cup_{i=1}^n U_i$ is open as a union of open sets and covers Y by construction. Furthermore, $V := \cap_{i=1}^n V_i$ is open as a finite union of open sets and it contains x_0 since x_0 is in each V_i. We also have that $U \cap V$ is empty, since any point of U is in some U_i, and thus not in V_i by construction, so it cannot be in V. □

Problem 74. Let X be a topological space. Show that X is compact if and only if for every collection of closed sets $\{C_\alpha\}_{\alpha \in \Lambda}$ that has the finite intersection property, we have $\cap_{\alpha \in \Lambda} C_\alpha \neq \{\}$.

Proof. ($\Rightarrow$) Suppose that X is compact but that there is a collection $\{C_\alpha\}_{\alpha \in \Lambda}$ of closed sets with the finite intersection property but has $\cap_{\alpha \in \Lambda} C_\alpha = \{\}$. Then we have that $X = \cup_{\alpha \in \Lambda}(X - C_\alpha)$, so $\{X - C_\alpha\}_{\alpha \in \Lambda}$ is an open cover of X. Since X is compact, there is a finite subcollection $\{X - C_i\}_{i=1}^n$ that covers X. But then $\cap_{i=1}^n C_i$ is empty, contrary to the assumption that the collection has the finite intersection property.

($\Leftarrow$) Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be an open cover of X. Then we have that $\{X - U_\alpha\}_{\alpha \in \Lambda}$ is a collection of closed sets such that $\cap_{\alpha \in \Lambda}(X - U_\alpha) = \{\}$, for otherwise there would be

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a point not covered by any of the U_α. Since this intersection is empty, the collection $\{X - U_\alpha\}_{\alpha\in\Lambda}$ cannot have the finite intersection property. Thus, there is a finite subcollection $\{X - U_i\}_{i=1}^n$ such that $\cap_{i=1}^n (X - U_i) = \{\}$, meaning that $X = \cup_{i=1}^n U_i$. Thus, since our initial collection was arbitrary, X is compact. □

Problem 75. Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies for X. Assume that X is compact and Hausdorff for both topologies and that $\mathcal{T} \subseteq \mathcal{T}'$. Show that $\mathcal{T} = \mathcal{T}'$.

Proof. Let $U \in \mathcal{T}'$. Then $X - U$ is closed under $\mathcal{T}'$, so the compactness of X implies that $X - U$ is compact for $\mathcal{T}'$. Now let $\{U_\alpha\}_{\alpha\in\Lambda}$ be a covering of $X - U$ by sets of $\mathcal{T}$. Since $\mathcal{T} \subseteq \mathcal{T}'$, this is also a covering by open sets of the topology $\mathcal{T}'$. Now we can use the compactness of $X - U$ to find a finite subcover $X - U \subseteq \cup_{i=1}^n U_i$. But each U_i is in $\mathcal{T}$, so this is a finite subcover stricting within $\mathcal{T}$, so $X - U$ is compact for $\mathcal{T}$. But X is Hausdorff for $\mathcal{T}$, so $X - U$ is closed, and thus $U \in \mathcal{T}$ as desired. □

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