Problem 2 (1/17/03): Let $A \in \mathbb{R}^{n \times n}$. Prove that

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|.$$  \hspace{1cm} (1)

Solution: Let $x \in \mathbb{R}^n$.

$$|[Ax]_i| = \left| \sum_{j=1}^{n} a_{ij}x_j \right| \leq \sum_{j=1}^{n} |a_{ij}| |x_j|.$$  

Since $|x_j| \leq \|x\|_\infty$ for all $j$, we have

$$|[Ax]_i| \leq \|x\|_\infty \sum_{i=1}^{n} |a_{ij}| \leq \|x\|_\infty \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|.$$  

Thus, maximizing the left hand side of the last equation over $i$ and dividing by $\|x\|_\infty$ we obtain

$$\frac{\|Ax\|_\infty}{\|x\|_\infty} \leq \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|.$$  

Maximizing the left hand side over all $x \neq 0$ we get

$$\|A\|_\infty \leq \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|.$$  

Let $\ell$ such that

$$\sum_{j=1}^{n} |a_{\ell j}| = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|.$$  

Choose $x$ as follows:

$$x_i = \begin{cases} 1 & \text{if } a_{\ell j} \geq 0 \\ -1 & \text{otherwise.} \end{cases}$$  

We have that $\|x\|_\infty = 1$ and $[Ax]_\ell = \sum_{j=1}^{n} |a_{\ell j}|$. Thus

$$\|A\|_\infty \geq \|Ax\|_\infty \geq [Ax]_\ell = \sum_{j=1}^{n} |a_{\ell j}| = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|,$$

which concludes the proof.

Problem 4 (1/17/03): Let $A, B \in \mathbb{R}^{n \times n}$. Prove that $\|AB\| \leq \|A\| \cdot \|B\|$.
Solution: Let $x \in \mathbb{R}^n$. Let $y = Bx$ and $C = AB$. Then $\|Cx\| = \|ABx\| = \|Ay\| \leq \|A\| \|y\| = \|A\| \|Bx\| \leq \|A\| \|B\| \|x\|$. Thus

$$\frac{\|Cx\|}{\|x\|} \leq \|A\| \|B\|.$$

Maximizing the left hand side of the last equation over all vectors different from 0 we obtain the desired result.

Problem 5 (1/17/03): Let $A \in \mathbb{R}^{n \times n}$. Prove that $\kappa(A) \geq 1$ ($\kappa(A)$ is the condition number of $A$).

Solution: $I = AA^{-1}$. Thus, from the previous exercise $\|I\| \leq \|A\| \|A^{-1}\| = \kappa(A)$. Since $\|I\| = 1$, we obtain $\kappa(A) \geq 1.$