P1) $\frac{y'}{x} + 2y = 1$

$$y' + 2xy = x$$

$$I = e^{\int 2x\,dx} = e^{x^2}$$

$$e^{x^2} y' + 2x e^{x^2} y = x e^{x^2}$$

$$\left(e^{x^2} y\right)' = x e^{x^2}$$

$$e^{x^2} y = \int x e^{x^2}\,dx$$

$$e^{x^2} y = \frac{1}{2} e^{x^2} + C$$

$$y = \frac{1}{2} + C e^{-x^2} \qquad y(1) = 1$$

$$1 = \frac{1}{2} + C e^{-1} \qquad C = \frac{e}{2}$$

$$\boxed{y = \frac{1}{2} + \frac{e^{1-x^2}}{2}}$$

P2) $y' = \frac{x}{y}$

$$\int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C$$

$x = 4 \quad y = 5$

$$\frac{25}{2} = \frac{16}{2} + C$$

$$C = \frac{9}{2}$$

$$y^2 = x^2 + 9$$

$$\boxed{y = \sqrt{9 + x^2}}$$

P3) $y'' + 2y' + y = e^{-x}$

$P(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2 = 0$

$\lambda = -1$ | $y = x^2 A e^{-x}$

$k = 2$ | $y' = A2xe^{-x} - Ax^2e^{-x}$

$l = 0$ | $y'' = 2Ae^{-x} - 4xAe^{-x} + Ax^2e^{-x}$

$y'' + 2y' + y = Ae^{-x}(2 - 4x + x^2) +$

$2Ae^{-x}(2x - x^2) + x^2Ae^{-x} =$

$= Ae^{-x}\,2 = e^{-x} \Rightarrow A = \frac{1}{2}$

$y_p = \frac{1}{2}x^2e^{-x}$

$y = \frac{1}{2}x^2e^{-x} + c_1e^{-x} + c_2xe^{-x}$

$y' = xe^{-x} - \frac{1}{2}x^2e^{-x} - c_1e^{-x} + c_2e^{-x}$

$- c_2xe^{-x}$

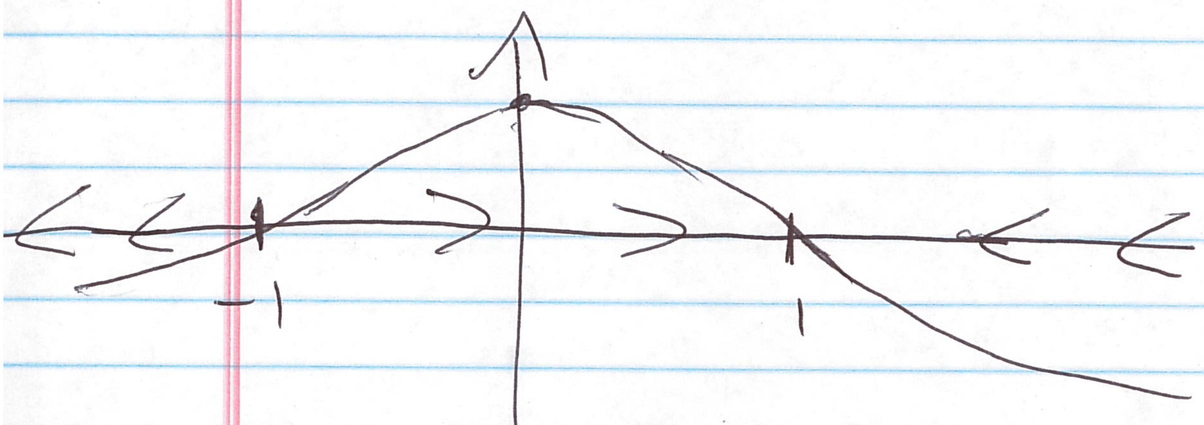
$\boxed{y = \frac{1}{2}x^2e^{-x} + e^{-x} + 2xe^{-x}}$

$y(0) = c_1 = 1$

$y'(0) = -c_1 + c_2 = 1$

P4) $\dot{x} = f(x) \qquad f(x) = e^x\left(\frac{2}{1+x^2} - 1\right)$

$f(x) = 0 \Rightarrow \frac{2}{1+x^2} - 1 = 0 \Rightarrow x = \pm 1$



Equilibrium points	stability
-1	unstable
1	stable

P5) $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$$P(\lambda) = (1-\lambda)^2 + 4 = 0$$

$$\lambda = 1 \pm 2i$$

$\lambda = 1+2i \qquad A - \lambda I = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \qquad v = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$$\begin{bmatrix} 1 \\ i \end{bmatrix} e^{(1+2i)t} = \begin{bmatrix} e^t \cos 2t \\ -e^t \sin 2t \end{bmatrix} + i \begin{bmatrix} e^t \sin 2t \\ e^t \cos 2t \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} e^t \cos 2t \\ -e^t \sin 2t \end{bmatrix} + c_2 \begin{bmatrix} e^t \sin 2t \\ e^t \cos 2t \end{bmatrix}$$

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e^t (\cos 2t + \sin 2t) \\ e^t (\cos 2t - \sin 2t) \end{bmatrix}$$

P6) $xy + y = 0$

$-y^2 + 3x - 3y + xy = 0$

$y(x+1) = 0 \Rightarrow y = 0$ or $x = -1$

$y = 0 \Rightarrow 3x = 0 \Rightarrow x = 0$ $\quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x = -1 \Rightarrow -y^2 - 3 - 3y \mp y = 0$

$y^2 + \cancel{2}4y + 3 = 0$ $\quad$ ~~$y=-3$ or $y = 1$~~

~~$\frac{-2 \pm \sqrt{4 - 4 \cdot 12}}{2}$~~ ~~no root~~ $\quad y = -1$ or $y = -3$

$$DF = \begin{bmatrix} y & x+1 \\ 3+y & -2y-3+x \end{bmatrix}$$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $DF(0,0) = \begin{bmatrix} 0 & 1 \\ 3 & -3 \end{bmatrix}$

$P(\lambda) = (-\lambda)(-3-\lambda) - 3 = \lambda^2 + 3\lambda - 3 = 0$

$\frac{-3 \pm \sqrt{9+12}}{2}$ $\quad$ unstable

~~Equilibrium points: only $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ unstable~~

$\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ $DF\left(\begin{bmatrix} -1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 2 & -2 \end{bmatrix}$ stable

$\begin{bmatrix} -1 \\ -3 \end{bmatrix}$ $DF\left(\begin{bmatrix} -1 \\ -3 \end{bmatrix}\right) = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$ unstable

Fixed points	Stability
$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	unstable
$\begin{bmatrix} -1 \\ -1 \end{bmatrix}$	stable
$\begin{bmatrix} -1 \\ -3 \end{bmatrix}$	unstable