Stability of fixed points

$\dot{x} = F(x) \quad x \in \mathbb{R}^n \quad x = x(t) \quad F: \mathbb{R}^n \to \mathbb{R}^n$

Let x_0 be a fixed point of $\dot{x} = F(x)$, i.e. $F(x_0) = 0$

Let x be close to x_0, i.e. $\|x - x_0\|$ is small

$$\boxed{F(x) \simeq \underbrace{F(x_0)}_{=0} + DF(x_0)(x - x_0) = DF(x_0)(x - x_0)}$$

$$DF = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

$F = (f_1, f_2, \ldots, f_n)$

Let $z = x - x_0$, $x = x(t)$. Let $A = DF(x_0)$

$$\boxed{\dot{z} = \dot{x} \simeq DF(x_0)(x - x_0) = DF(x_0)\, z}$$ i.e.

$$\boxed{\dot{z} = A z}$$ with $A = DF(x_0)$

<u>Theorem</u>: Let x_0 be a fixed point of $\dot{x} = F(x)$. Then:

1) x_0 is stable if all the eigenvalues of $DF(x_0)$ have negative real part

2) If one of the eigenvalues of $DF(x_0)$ has positive real part, then x_0 is unstable

3) Otherwise, we do not know if x_0 is stable or unstable

Example: Find the fixed points and study their stability

1) $\dot{x} = 1 - 2xy$ $\qquad\qquad$ $0 = 1 - 2xy$

$\dot{y} = 2xy - y$ $\qquad\qquad$ $0 = 2xy - y \rightarrow 0 = (2x-1)y \rightarrow$

$\rightarrow y = 0$ or $x = \frac{1}{2}$

$y = 0 \Rightarrow 1^{st}$ eq implies $0 = 1$ impossible

$x = \frac{1}{2} \Rightarrow 1^{st}$ eq implies $y = 1$

fixed points: $\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

$$DF = \begin{bmatrix} -2y & -2x \\ 2y & 2x-1 \end{bmatrix} \qquad DF\left(\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 & -1 \\ 2 & 0 \end{bmatrix}$$

Eigenvalues of $DF\left(\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\right)$

$$\det \begin{bmatrix} -2-\lambda & -1 \\ \end{bmatrix} = (-2-\lambda)(-\lambda) + 2 = \lambda^2 + 2\lambda + 2 = 0$$

$$\det\begin{bmatrix}-2-\lambda & -1\\ 2 & -\lambda\end{bmatrix} = (-2-\lambda)(-\lambda)+2 = \lambda^2+2\lambda+2 = 0$$

$\lambda = -1 \pm i$ $\quad Re(\lambda) = -1 < 0$. then $\begin{bmatrix}1/2\\1\end{bmatrix}$ is stable

2) $\dot{x} = -3x + y^2 + 2$

$\dot{y} = x^2 - y^2$

Find fixed points

$$\left.\begin{aligned}0 &= -3x + y^2 + 2\\ 0 &= x^2 - y^2 \longrightarrow y^2 = x^2\end{aligned}\right\} \longrightarrow 0 = -3x + x^2 + 2 = (x-1)(x-2)$$

$x = 1$ then $y^2 = x^2 \Rightarrow y = \pm 1$

$x = 2$ then $y^2 = x^2 \Rightarrow y = \pm 2$

Fixed points: $\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\-2\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}$

$$DF = \begin{bmatrix}-3 & 2y\\ 2x & -2y\end{bmatrix}$$

$\begin{bmatrix}1\\-1\end{bmatrix}$ $DF\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}-3 & -2\\ 2 & 2\end{bmatrix}$ $\quad P(\lambda) = (-3-\lambda)(2-\lambda) + 4 =$

$= \lambda^2 + \lambda - 2 = (\lambda+2)(\lambda-1)$

↑ unstable

$\begin{bmatrix}1\\1\end{bmatrix}$ $DF\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}-3 & 2\\ 2 & -2\end{bmatrix}$ $\quad P(\lambda) = (-3-\lambda)(-2-\lambda) - 4 = \lambda^2 + 5\lambda + 2 = 0$

$\lambda = \dfrac{-5 \pm \sqrt{25-8}}{2} < 0$

↖ stable

$\begin{bmatrix}2\\-2\end{bmatrix}$ $DF\left(\begin{bmatrix}2\\-2\end{bmatrix}\right) = \begin{bmatrix}-3 & -4\\ \cdot & \end{bmatrix}$ $\quad P(\lambda) = (-3-\lambda)(4-\lambda) + 16 =$

$\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ $DF\left(\begin{bmatrix} 2 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} -3 & -4 \\ 4 & 4 \end{bmatrix}$ $P(\lambda) = (-3-\lambda)(4-\lambda)+16 =$

$= \lambda^2 - \lambda + 4 = 0$

unstable

$$\lambda = \frac{1 \pm \sqrt{1-16}}{2} = \frac{1 \pm i\sqrt{15}}{2} \quad Re>0$$

$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $DF\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -3 & 4 \\ 4 & -4 \end{bmatrix}$ $P(\lambda) = (-3-\lambda)(-4-\lambda)-16 =$

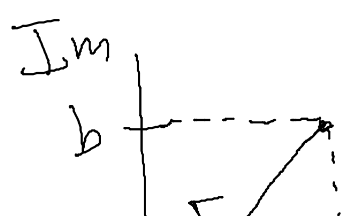
$= \lambda^2 + 7\lambda - 4 = 0$

unstable

$$\lambda = \frac{-7 \pm \sqrt{49+16}}{2} \qquad \frac{-7+\sqrt{65}}{2} > 0$$

Fixed point	stability
$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$	unstable
$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	stable
$\begin{bmatrix} 2 \\ -2 \end{bmatrix}$	unstable
$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$	unstable

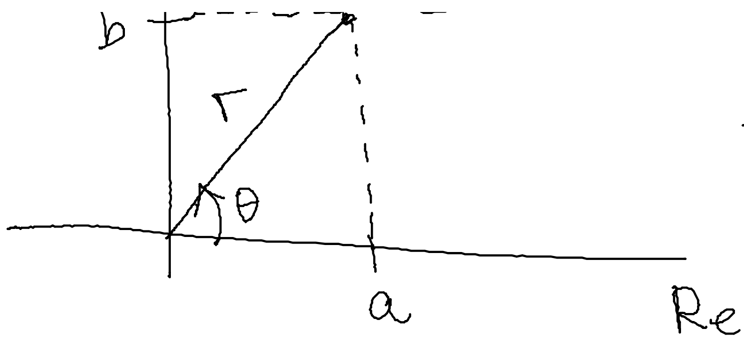
Complex numbers



$z = a + bi$

$\theta = \arg(z)$

$r = |z| = \sqrt{a^2+b^2}$



$r = |z| = \sqrt{a+b}$

$\tan\theta = \dfrac{b}{a}$

$a = \mathrm{Re}(z) \qquad b = \mathrm{Im}(z)$

$a = r\cos\theta \qquad b = r\sin\theta$

$z = r(\cos\theta + i\sin\theta) = r\,e^{i\theta}$ polar form of z

$z = a+bi$ cartesian form

<u>Obs</u>: $z = re^{i\theta} \qquad w = \rho e^{i\beta}$

$$\boxed{zw = r\rho\, e^{i(\theta+\beta)}}$$

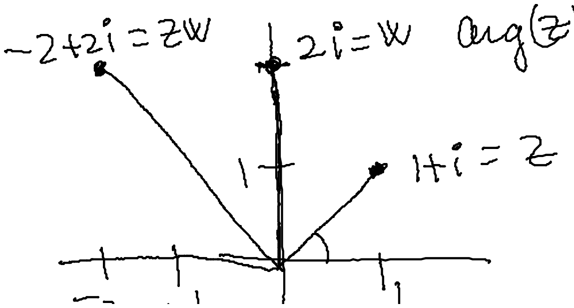
The length of the product is the product of the lengths

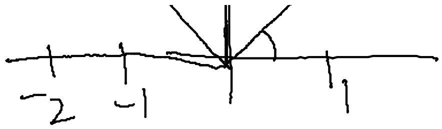
the argument of the product is the sum of the arguments

<u>Example</u> $z = 1+i \qquad w = 2i \qquad zw = -2+2i$

$|z| = \sqrt{2} \qquad |w| = 2 \qquad |zw| = \sqrt{8} = 2\sqrt{2}$

$\arg(z) = \frac{\pi}{4} \qquad \arg w = \frac{\pi}{2} \qquad \arg(zw) = \frac{3\pi}{4}$





Obs: $z = re^{i\theta}$ then $z^{-1} = \frac{1}{r} e^{-i\theta}$

$|z^{-1}| = |z|^{-1} = \frac{1}{|z|}$ $\qquad$ $\arg(z^{-1}) = -\arg(z)$

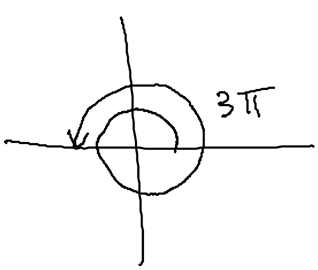
Obs: $z = re^{i\theta}$ and $w = \rho e^{i\beta}$

$\frac{z}{w} = \frac{r}{\rho} e^{i(\theta-\beta)}$ $\qquad$ $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$ $\qquad$ $\arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w)$

Obs: $z = re^{i\theta}$ then $z^n = r^n e^{in\theta}$

Example: $(2i)^6$ $\qquad$ $2i = 2e^{\frac{i\pi}{2}}$

$$\boxed{(2i)^6 = 2^6 e^{\frac{i6\pi}{2}} = 64 \underbrace{e^{i3\pi}}_{=-1} = -64}$$



Roots of complex numbers

Let $z \in \mathbb{C}$. Let n be a positive integer. The n^{th} roots of z are the complex numbers w such that $w^n = z$

$z = re^{i\theta}$ given

$w = \rho e^{i\beta}$ unknown

$w^n = z \qquad \rho^n e^{in\beta} = re^{i\theta}$

then $\rho^n = r$, then $\boxed{\rho = r^{1/n}}$

$n\beta = \theta + 2\pi k$ $\quad k$ integer

$$\boxed{\beta = \beta_k = \frac{\theta}{n} + \frac{2\pi k}{n} \qquad 0 \le k \le n-1}$$

the n^{th} roots of z are $w_k = \rho e^{i\beta_k}$ where

$\rho = r^{1/n} = |z|^{1/n}$ and $\beta_k = \frac{\theta}{n} + \frac{2\pi k}{n} = \frac{\arg(z)}{n} + \frac{2\pi k}{n}$

$0 \le k \le n-1$

Example: Find the cubic roots of 8

$z = 8 \qquad w^3 = 8$

$r = 8$

$\theta = \arg(z) = \arg(8) = 0$

$\rho = 8^{1/3} = 2$

$\beta_k = \frac{0}{3} + \frac{2\pi k}{3} \qquad 0 \le k \le 2$

3 solutions

$w = w_0, w_1$ & w_2

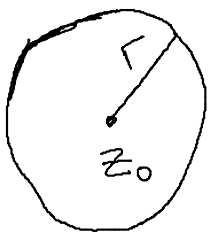
$w_0 = 2\, e^{i0} = \boxed{2}$

$w_1 = 2\, e^{i\frac{2\pi}{3}} = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = \boxed{-1 + i\sqrt{3}}$

$w_2 = 2\, e^{i\frac{4\pi}{3}} = 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = \boxed{-1 - i\sqrt{3}}$

$$w_2 = 2\, e^{i\frac{4\pi}{3}} = 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = \boxed{-1 - i\sqrt{3}}$$

Sets in the complex plane



z_0 given $z_0 \in \mathbb{C}$

r given $r > 0$

$|z - z_0| = r$ circle of radius r centered at z_0

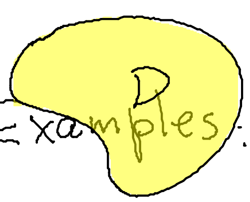
$|z - z_0| < r$ open disk of radius r centered at z_0

Open sets: A set D in the complex plane is said to be open if it does not contain any of the points in its boundary.

Example: All the open disks are open sets.

$\mathbb{C}$ is an open set

Closed sets: A set D is closed if it contains all its boundary points



Examples: 1) $|z - 2i| \leq 3$ is closed

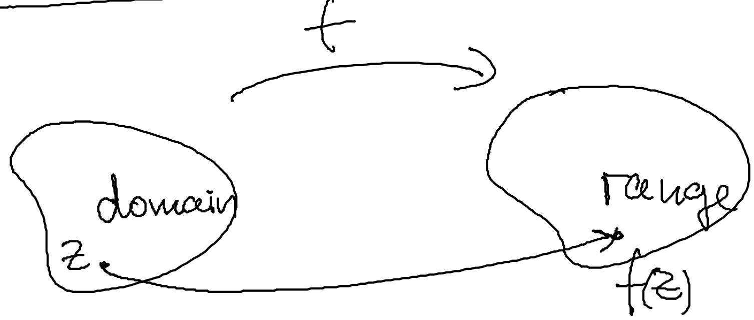
D is neither open nor closed

2) $\mathbb{C}$ is both open and closed

3)

Def: A neighborhood of a point z_0 is an open disk centered at z_0

Functions of complex variable



The domain and the range are both sets of complex numbers.

Example: $f(z) = z + 2z^2$

$$\boxed{f(1+i) = 1+i+2(1+i)^2 = 1+i+2\,2i = 1+5i}$$

Obs: $z = x+iy \quad x,y \in \mathbb{R}$

$f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$

$u : \mathbb{R}^2 \to \mathbb{R} \quad \& \quad v : \mathbb{R}^2 \to \mathbb{R}$

x, y are real.
u & v are real valued functions

Example: $f(z) = z + 2z^2$

$z = x + iy \qquad z^2 = x^2 - y^2 + 2ixy$

$f(z) = x + iy + 2(x^2 - y^2 + 2ixy)$

$f(z) = (x + 2x^2 - 2y^2) + i(y + 4xy)$

$u(x,y) = x + 2x^2 - 2y^2$

$v(x,y) = y + 4xy$