<u>Obs</u>: f has a finite number of singularities in $\mathrm{Im}(z) \geq 0$. Let

$$M_R = \max_{\substack{|z| = R \\ \mathrm{Im}(z) \geq 0}} |f(z)|$$

Assume that $\lim_{R \to \infty} M_R = 0$.

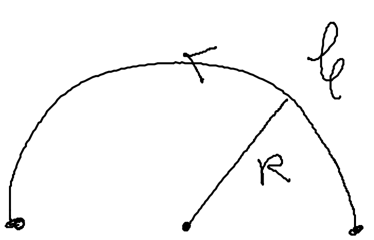
Then $$\lim_{R \to \infty} \int_{\substack{|z|=R \\ \mathrm{Im}(z) \geq 0}} e^{iaz} f(z)\,dz = 0$$

for any a such that $a > 0$

$$\boxed{\begin{array}{ll} z = x+iy & e^{iaz} = e^{-ay}\, e^{iax} \\ \mathrm{Im}\, z \geq 0 \quad y \geq 0 & \end{array}}$$

<u>Example</u>: $$\lim_{R \to \infty} \int_{\substack{|z|=R \\ \mathrm{Im}(z) \geq 0}} e^{iz} \frac{1}{z}\, dz = 0$$

$f(z) = \frac{1}{z}$



$$M_R = \max_{\substack{|z|=R \\ \mathrm{Im}(z) \geq 0}} |f(z)| = \frac{1}{R} \longrightarrow 0 \text{ as } R \to \infty$$

$a = 1$

<u>Obs</u>. Assume f has a finite number of singu-

$\int e^{iaz} f(z)dz = 0$

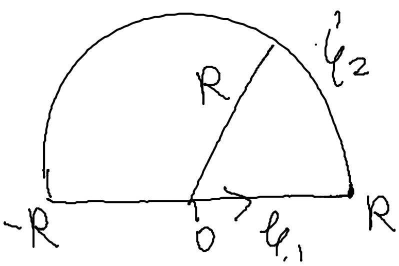
larities in $\mathrm{Im}(z) \geq 0$. Assume $\displaystyle\lim_{\substack{R\to\infty}} \int_{\substack{|z|=R \\ \mathrm{Im}(z)\geq 0}} e^{iaz} f(z)\,dz = 0$

$a > 0$.

Assume f has no singularities in the real axis. Then

$$\int_{-\infty}^{\infty} f(x)\, e^{iax}\, dx = 2\pi i \sum_{\substack{z_k \text{ singularity of } f(z) \\ \mathrm{Im}(z_k) > 0}} \mathrm{Res}\left(f(z) e^{iaz}, z_k\right)$$

proof:



$$\int_{\mathcal{C}_1} f(z)\, e^{iaz}\, dz + \int_{\mathcal{C}_2} f(z)\, e^{iaz}\, dz = $$

$$= 2\pi i \sum_{\substack{z_k \text{ singularity of } f \\ \text{inside } \mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2}} \mathrm{Res}\left(f(z) e^{iaz}, z_k\right)$$

take $\displaystyle\lim_{R\to\infty}$ to get $\displaystyle\int_{\mathcal{C}_2} f(z)\, e^{iaz}\, dz \longrightarrow 0$ (from last Obs)

$$\int_{\mathcal{C}_1} f(z) e^{iaz}\, dz \longrightarrow \int_{-\infty}^{\infty} f(x)\, e^{iax}\, dx$$

Obs: $\displaystyle\int_{-\infty}^{\infty} f(x) \sin ax\, dx = \mathrm{Im}\left(\int_{-\infty}^{\infty} f(x)\, e^{iax}\, dx\right)$

$$\int_{-\infty}^{\infty} f(x) \cos ax\, dx = \mathrm{Re}\left(\int_{-\infty}^{\infty} f(x)\, e^{iax}\, dx\right)$$

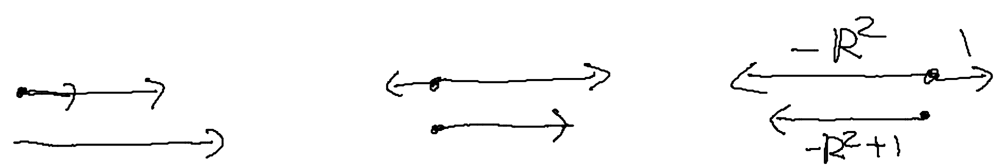
<u>Example:</u> $\displaystyle\int_{-\infty}^{\infty} \frac{x}{x^2+1} \sin 2x \; dx$

$f(z) = \dfrac{z}{z^2+1}$ $\qquad \displaystyle\int_{-\infty}^{\infty} f(x)\, e^{iax}\; dx$

$a = 2$ 1) $a > 0$ ✓ 2) Singularities of $f(z)$ in $\mathrm{Im}(z) > 0$ only $z = i$ ✓ 3) f has no singularitis in Real axis

4) $M_R = \max\limits_{\substack{\mathrm{Im}(z) \geq 0 \\ |z| = R}} |f(z)| \longrightarrow 0$

$f(z) = \dfrac{z}{z^2+1}$ $\qquad |f(z)| = \dfrac{|z|}{|z^2+1|} \leq \dfrac{R}{R^2-1} \approx \dfrac{1}{R} \to 0$ as $R \to \infty$



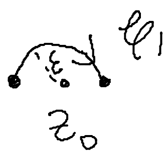
$z = iR$

then $\displaystyle\int_{-\infty}^{\infty} \frac{x}{x^2+1} \sin(2x)\, dx = \mathrm{Im}\left[2\pi i \,\mathrm{Res}\left(\frac{z}{z^2+1} e^{i2z}, \; i\right)\right]$

$\mathrm{Res}\left(\dfrac{z}{z^2+1} e^{i2z}, i\right) = \lim\limits_{z \to i} (z-i) \dfrac{z}{(z^2+1)} e^{i2z} = \dfrac{i\, e^{-2}}{2i} = \dfrac{e^{-2}}{2}$

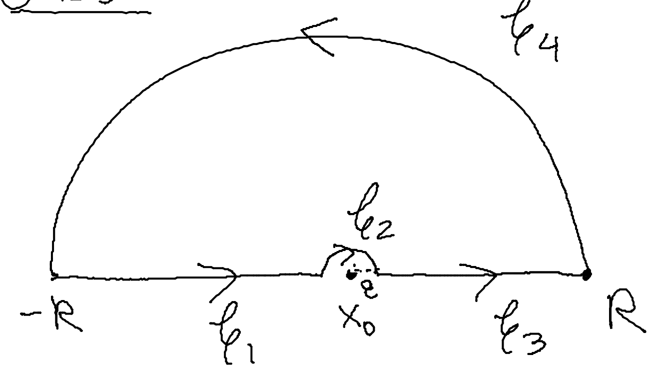
$z^2+1 = (z+i)(z-i)$

$$\int_{-\infty}^{\infty} \frac{x}{x^2+1} \sin 2x \, dx = \operatorname{Im}\left(\frac{2\pi i \, e^{-2}}{2} \right) = \boxed{\pi e^{-2}}$$

Obs: 

$$\boxed{\int_{\ell_1} \frac{dz}{z-z_0} = \int_{-\pi}^{0} \frac{\varepsilon(-i)\, e^{it}\, dt}{\varepsilon e^{-it}} = -i\pi}$$

Obs



1) x_0 is the only singularity of f in the real axis.

2) x_0 is a simple pole.

3) $\displaystyle \lim_{R\to\infty} \int_{\substack{|z|=R \\ \operatorname{Im}(z)>0}} f(z)\, dz = 0$

4) Assume f has a finite number of singularities in $\operatorname{Im}(z) > 0$. then

$$\boxed{\int_{-\infty}^{\infty} f(x)\, dx = +i\pi \operatorname{Res}(f(z), x_0) + 2\pi i \sum_{\substack{z_k \text{ singularity of } f \\ \operatorname{Im}(z_k)>0}} \operatorname{Res}(f(z), z_k)}$$

proof: $\displaystyle \lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \left(\int_{\ell_1} f(z)dz + \int_{\ell_2} f(z)\, dz + \int_{\ell_3} f(z)\, dz + \int_{\ell_4} f(z)\, dz \right) =$

$$= 2\pi i \sum_{\operatorname{Im}(z_k)>0} \operatorname{Res}(f(z), z_k) .$$

z_k singularity of $f(z)$

$$\lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \int_{\mathcal{C}_1} f(z)\,dz + \int_{\mathcal{C}_3} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx$$

$$\lim_{R\to\infty} \int_{\mathcal{C}_4} f(z)\,dz = 0 \qquad a_{-1} = \operatorname{Res}(f(z), x_0)$$

$$f(z) = \frac{a_{-1}}{z-x_0} + g(z) \qquad g(z) \text{ analytic at } x_0$$

$$\lim_{\varepsilon\to 0} \int_{\mathcal{C}_2} f(z)\,dz = \underbrace{\lim_{\varepsilon\to 0} \int_{\mathcal{C}_2} \frac{a_{-1}\,dz}{z-x_0}}_{-i\pi a_{-1}} + \underbrace{\lim_{\varepsilon\to 0} \int_{\mathcal{C}_2} g(z)\,dz}_{\to 0}$$

<u>Example</u>: Compute $\displaystyle\int_0^{\infty} \frac{\sin x}{x}\,dx$

1) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = 2\int_0^{\infty} \frac{\sin x}{x}\,dx$

2) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \operatorname{Im}\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx\right)$

$f(z) = \dfrac{e^{iz}}{z}$ a) no singularities in $\operatorname{Im}(z) > 0$
b) $z_0 = 0$ is the only singularity in the real axis

c) $\displaystyle\lim_{R\to\infty} \int_{|z|=R} \frac{e^{iz}}{z}\,dz = 0$ because $M_R = \max_{\substack{|z|=R\\ \operatorname{Im}(z)\geq 0}} \left|\frac{1}{z}\right| = \frac{1}{R} \underset{R\to\infty}{\longrightarrow} 0$

$\mathrm{Im}(z)>0$

then $\displaystyle\int_{\substack{|z|=R \\ \mathrm{Im}(z)>0}} \frac{1}{z} e^{iaz}\, dz \xrightarrow[R\to\infty]{} 0$

$1=a>0$

Then $\displaystyle\int_{-\infty}^{\infty} \frac{e^{iz}}{z}\, dz = i\pi \,\mathrm{Res}\left(\frac{e^{iz}}{z}, 0\right) = i\pi$

then $\displaystyle\int_{0}^{\infty} \frac{\sin x}{x}\, dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \frac{1}{2}\,\mathrm{Im}(i\pi) = \boxed{\frac{\pi}{2}}$