Note Title 5/27/2015

Solving linear systems

Def: Elementary operations (change the augmented matrix without changing the set of solutions):

1) Multiply a row by a non-zero number
2) Interchange two rows
3) Add to a row a multiple of an other row

Gaussian elimination

Do elementary operations to change the augmented matrix into RRE form

Example

$$2x_1 + 6x_2 + x_3 = 7$$
$$x_1 + 2x_2 - x_3 = -1$$
$$5x_1 + 7x_2 - 4x_3 = 9$$

First write down the augmented matrix

$$\left[\begin{array}{ccc|c} \boxed{2} & 6 & 1 & 7 \\ 1 & 2 & -1 & -1 \\ 5 & 7 & -4 & 9 \end{array}\right]$$

$$R_1/2 \quad \left[\begin{array}{ccc|c} \boxed{1} & 3 & 1/2 & 7/2 \\ 1 & 2 & -1 & -1 \\ 5 & 7 & -4 & 9 \end{array}\right]$$

$$\begin{array}{c} \\ R_2 - R_1 \\ R_3 - 5R_1 \end{array} \left[\begin{array}{ccc|c} \boxed{1} & 3 & 1/2 & 7/2 \\ 0 & \boxed{-1} & -3/2 & -9/2 \\ 0 & -8 & -13/2 & -17/2 \end{array}\right]$$

$$\begin{array}{c} \\ R_2/(-1) \\ \\ \end{array} \left[\begin{array}{ccc|c} \boxed{1} & 3 & 1/2 & 7/2 \\ 0 & \boxed{1} & 3/2 & 9/2 \\ 0 & -8 & -13/2 & -17/2 \end{array}\right]$$

$$R_3+8R_2 \left[\begin{array}{ccc|c} \boxed{1} & 3 & 1/2 & 7/2 \\ 0 & \boxed{1} & 3/2 & 9/2 \\ 0 & 0 & \boxed{11/2} & 55/2 \end{array}\right]$$

$$R_3/(11/2) \left[\begin{array}{ccc|c} \boxed{1} & 3 & 1/2 & 7/2 \\ 0 & \boxed{1} & 3/2 & 9/2 \\ 0 & 0 & \boxed{1} & 5 \end{array}\right]$$

$$\begin{array}{l} R_1-\frac{1}{2}R_3 \\ R_2-\frac{3}{2}R_3 \end{array} \left[\begin{array}{ccc|c} \boxed{1} & 3 & 0 & 1 \\ 0 & \boxed{1} & 0 & -3 \\ 0 & 0 & \boxed{1} & 5 \end{array}\right]$$

$$R_1-3R_2 \left[\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$\boxed{\begin{array}{l} x_1 = 10 \\ x_2 = -3 \\ x_3 = 5 \end{array}} \qquad X = \begin{bmatrix} 10 \\ -3 \\ 5 \end{bmatrix}$$

Example

$$\begin{aligned} x_1 + x_2 - x_3 + x_4 &= 0 \\ 2x_1 + 2x_2 + x_3 + x_4 &= 1 \\ x_3 + 2x_4 &= -1 \end{aligned}$$

$$\left[\begin{array}{cccc|c} \boxed{1} & 1 & -1 & 1 & 0 \\ 2 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 \end{array}\right]$$

$$R_2-2R_1 \left[\begin{array}{cccc|c} \boxed{1} & 1 & -1 & 1 & 0 \\ 0 & 0 & \boxed{3} & -1 & 1 \\ 0 & 0 & 1 & 2 & -1 \end{array}\right]$$

$$R_2/3 \quad \left[\begin{array}{cccc|c} \boxed{1} & 1 & -1 & 1 & 0 \\ 0 & 0 & \boxed{1} & -1/3 & 1/3 \\ 0 & 0 & 1 & 2 & -1 \end{array}\right]$$

$$\begin{array}{l} \\ \\ R_3-R_2 \end{array} \left[\begin{array}{cccc|c} \boxed{1} & 1 & -1 & 1 & 0 \\ 0 & 0 & \boxed{1} & -1/3 & 1/3 \\ 0 & 0 & 0 & \boxed{7/3} & -4/3 \end{array}\right]$$

$$R_3/(7/3) \quad \left[\begin{array}{cccc|c} \boxed{1} & 1 & -1 & 1 & 0 \\ 0 & 0 & \boxed{1} & -1/3 & 1/3 \\ 0 & 0 & 0 & \boxed{1} & -4/7 \end{array}\right]$$

$$\begin{array}{l} R_1-R_3 \\ R_2+\frac{1}{3}R_3 \\ \end{array} \left[\begin{array}{cccc|c} \boxed{1} & 1 & -1 & 0 & 4/7 \\ 0 & 0 & \boxed{1} & 0 & 1/7 \\ 0 & 0 & 0 & \boxed{1} & -4/7 \end{array}\right]$$

$$R_1+R_2 \quad \left[\begin{array}{cccc|c} \boxed{1} & 1 & 0 & 0 & 5/7 \\ 0 & 0 & \boxed{1} & 0 & 1/7 \\ 0 & 0 & 0 & \boxed{1} & -4/7 \end{array}\right]$$

$$\boxed{\begin{aligned} x_1 &= 5/7 - t \\ x_2 &= t \\ x_3 &= 1/7 \\ x_4 &= -4/7 \end{aligned}}$$

$$\boxed{x = \begin{bmatrix} 5/7 \\ 0 \\ 1/7 \\ -4/7 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}$$

<u>Example</u>

$$\begin{aligned} x_1 + x_2 &= 1 \\ -x_1 - x_2 &= 2 \end{aligned}$$

$$\left[\begin{array}{cc|c} \boxed{1} & 1 & 1 \\ -1 & -1 & 2 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{cc|c} \boxed{1} & 1 & 1 \\ 0 & 0 & \boxed{3} \end{array}\right]$$

the last column is a pivot column, thus there are no solutions

Rank of a matrix

Def: The rank of a matrix A, denoted by $\text{rank}(A)$, is the maximum number of linearly independent rows in A

Example $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ 2 row 1 - row 2 = 0, then $\text{rank}(A) = 1$

Obs: 1) Doing Gaussian elimination on a matrix does not change its rank

2) If A is in RRE form, then $\text{rank}(A) = \#$ of pivots $=$
$= \#$ of leading ones

<u>Example:</u> Find rank(A), where $A = \begin{bmatrix} 1 & 1 & -1 & 3 \\ 2 & -2 & 6 & 8 \\ 3 & 5 & -7 & 8 \end{bmatrix}$

$$\begin{array}{c} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \begin{bmatrix} \boxed{1} & 1 & -1 & 3 \\ 0 & \boxed{-4} & 8 & 2 \\ 0 & 2 & -4 & -1 \end{bmatrix} \quad \begin{array}{c} \\ R_2(-4) \\ \\ \end{array} \begin{bmatrix} \boxed{1} & 1 & -1 & 3 \\ 0 & \boxed{1} & -2 & -1/2 \\ 0 & 2 & -4 & -1 \end{bmatrix} \quad \begin{array}{c} \\ \\ R_3 - 2R_2 \end{array} \begin{bmatrix} \boxed{1} & 1 & -1 & 3 \\ 0 & \boxed{1} & -2 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank$(A) = 2$

<u>Obs:</u> The system $Ax = b$ is consistent if and only if rank$(A) = $ rank$([A|b])$

<u>Obs:</u> Assume $Ax = b$ is consistent. $A = \mathbb{R}^{k \times n}$. Let $r = $ rank(A).

then:
1) $r = $ # of leading variables
2) $n - r = $ # of free variables
3) $n - r = $ # of parameters in the set of solutions

Determinants

Def: The determinant of a 2 by 2 matrix is

$$\det A = \det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = a_{11}a_{22} - a_{21}a_{12}$$

Def: $A \in \mathbb{R}^{n \times n}$. We denote by A_{ij} the $(n-1)$ by $(n-1)$ matrix that we get when we remove the i^{th} row and j^{th} column from the matrix A

Def: $A \in \mathbb{R}^{n \times n}$

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13} + \cdots + (-1)^{n+1} a_{1n} \det A_{1n}$$

Example $\det\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 2 & 4 & 1 \end{bmatrix} = 1\det\begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix} - (-1)\det\begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix} + 2\det\begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix}$

$= (-11) + (-6) + 2(-2) = -21$

Properties

1) $\det A^T = \det A$

2) $\det(AB) = \det A \det B$

3) If A is triangular, then $\det A = a_{11} a_{22} \cdots\cdots a_{nn}$

Inverse of matrices

Def: Let $A \in \mathbb{R}^{n\times n}$, we say that A has an inverse if there exists $B \in \mathbb{R}^{n\times n}$ such that $AB = BA = I$, where

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$ is the identity

B is called the inverse of A and it is denoted by $B = A^{-1}$

<u>Def</u>: $e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th}$ $\qquad e_i \in \mathbb{R}^n$

<u>Note</u>: $I = \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix}$ $\quad e_i$ is the i^{th} column of I.

<u>Obs</u>: $AB = I$ $\qquad B = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}$ $\quad b_i$ is the i^{th} column of B

$$AB = \begin{bmatrix} Ab_1 & Ab_2 & \cdots\cdots & Ab_n \end{bmatrix} = I = \begin{bmatrix} e_1 & e_2 & \cdots\cdots & e_n \end{bmatrix}$$

then $Ab_i = e_i$ for all $1 \le i \le n$

this is a linear system

To find $b_1, b_2, \ldots, b_n$, solve

$$\left[A \,\middle|\, e_1 \; e_2 \; \cdots\cdots \; e_n \right] = [A \mid I]$$

Do Gaussian elimination on $[A \mid I]$

Case 1: No free variables, then the RRE form we get after Gaussian elimination is

$$\left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$$

<u>Case 2</u>: there are some free variables. In this case A has no inverse

<u>Example</u> $A = \begin{bmatrix} 1 & 4 \\ 2 & 10 \end{bmatrix}$ Find A^{-1}

$$\left[\begin{array}{cc|cc} \boxed{1} & 4 & 1 & 0 \\ 2 & 10 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} \\ R_2 - 2R_1 \end{array} \left[\begin{array}{cc|cc} \boxed{1} & 4 & 1 & 0 \\ 0 & \boxed{2} & -2 & 1 \end{array}\right]$$

$$\begin{array}{c} \\ R_2/2 \end{array} \left[\begin{array}{cc|cc} \boxed{1} & 4 & 1 & 0 \\ 0 & \boxed{1} & -1 & 1/2 \end{array}\right]$$

$$R_1 - 4R_2 \quad \left[\begin{array}{cc|cc} 1 & 0 & 5 & -2 \\ 0 & 1 & -1 & 1/2 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 5 & -2 \\ -1 & 1/2 \end{bmatrix}$$

Check $\begin{bmatrix} 1 & 4 \\ 2 & 10 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -1 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

Facts: 1) $(A^{-1})^{-1} = A$

2) $(AB)^{-1} = B^{-1}A^{-1}$

3) $(A^{T})^{-1} = (A^{-1})^{T}$

4) A has an inverse if and only if $\det A \neq 0$

Def: We say that A is invertible or non-singular, if A has an inverse