Let $A \in \mathbb{R}^{n\times n}$. To diagonalize $A$ means to find $P \in \mathbb{R}^{n\times n}$ nonsingular and $D \in \mathbb{R}^{n\times n}$ diagonal such that $A = PDP^{-1}$

<u>Recipe</u>: Find the eigenvalues and eigenvectors of $A$.
If $A$ has $n$ linearly independent eigenvectors, then $A$ is diagonalizable. In this case $P = [v_1\ v_2 \dots v_n]$, where $v_1, \dots, v_n$ are $n$ linearly independent eigenvectors of $A$ and $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$, where $\lambda_1, \dots, \lambda_n$ are the eigenvalues, respecting the order, i.e. $Av_i = \lambda_i v_i$

If $A$ does not have $n$ linearly independent eigenvectors, then $A$ is not diagonalizable

<u>Example</u>: $A = \begin{bmatrix} -5 & 9 \\ -6 & 10 \end{bmatrix}$

$$P(\lambda) = \det \begin{bmatrix} -5-\lambda & 9 \\ -6 & 10-\lambda \end{bmatrix} = (-5-\lambda)(10-\lambda) + 54 = \lambda^2 - 5\lambda + 4 =$$
$$= (\lambda-4)(\lambda-1)$$

$\boxed{\lambda_1 = 1}$

$$A - I = \begin{bmatrix} -6 & 9 \\ -6 & 9 \end{bmatrix} \qquad \boxed{v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}}$$

$\boxed{\lambda_2 = 4}$

$$A - 4I = \begin{bmatrix} -9 & 9 \\ -6 & 6 \end{bmatrix} \qquad \boxed{v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$\boxed{P = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}} \quad \boxed{D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}}$$

$$\left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \Bigg| \left[\begin{array}{cc|cc} 1 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & -2/3 & 1 \end{array}\right] \Bigg| \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 3 \end{array}\right]$$

$$\boxed{P^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}} \qquad A = PDP^{-1}$$

Check $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 9 \\ -6 & 10 \end{bmatrix}$ ✓

<u>Reminder</u>: 1) $P \in \mathbb{R}^{n \times n}$ is orthogonal if $P^{-1} = P^T$

2) $P = [p_1\ p_2 \cdots p_n] \in \mathbb{R}^{n\times n}$ is orthogonal if and only if $p_1, p_2, \dots, p_n$ is orthonormal.

Def: We say that $A$ is orthogonally diagonalizable if there exists $P$ orthogonal and $D$ diagonal such that $A = PDP^T$ $(A, D, P \in \mathbb{R}^{n\times n})$

Obs: $A$ is orthogonally diagonalizable if and only if $A$ has $n$ eigenvectors that form an orthonormal set. if and only if $A$ is symmetric.

$A = PDP^T$ $P$ orthogonal $D$ diagonal.

$A^T = (P^T)^T D^T P^T = PDP^T = A$

Obs: To orthogonally diagonalize a symmetric matrix $A \in \mathbb{R}^{n\times n}$:

1st: Find $\lambda_1, \dots, \lambda_r$ the eigenvalues (different) of $A$

$\lambda_i \neq \lambda_j$ if $i \neq j$

2nd: For each $1 \leq i \leq r$, find $v_1^{(i)}, \dots, v_{n_i}^{(i)}$ linearly inde pendent eigenvectors with eigenvalue $\lambda_i$, where $n_i$ is the multiplicity of $\lambda_i$. Apply Gram-Schmidt to $v_1^{(i)}, \dots, v_{n_i}^{(i)}$ to get $u_1^{(i)}, \dots, u_{n_i}^{(i)}$

Then
$$P = \begin{bmatrix} u_1^{(1)} \cdots u_{n_1}^{(1)} & u_1^{(2)} \cdots u_{n_2}^{(2)} & \cdots\cdots & u_1^{(r)} \cdots u_{n_r}^{(r)} \end{bmatrix}$$

$$D = \begin{bmatrix} \underbrace{\lambda_1 \cdots \lambda_1}_{n_1} & & & \\ & \underbrace{\lambda_2 \cdots \lambda_2}_{n_2} & & 0 \\ & & \ddots & \\ 0 & & & \underbrace{\lambda_r \cdots \lambda_r}_{n_r} \end{bmatrix}$$

<u>Least square problem</u>

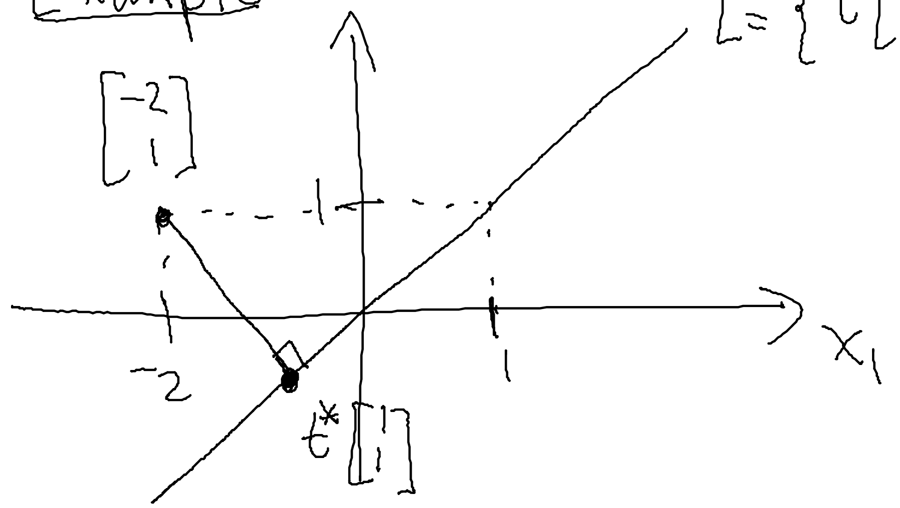
$$Ax = b \Rightarrow Ax - b = 0$$

<u>Def</u>: The least square problem $Ax=b$ is minimizing $\|Ax-b\|$

$\|Ax-b\|$

**Example**

$L = \{ t\begin{bmatrix}1\\1\end{bmatrix} : t \in \mathbb{R} \} = \operatorname{span}\{\begin{bmatrix}1\\1\end{bmatrix}\}$

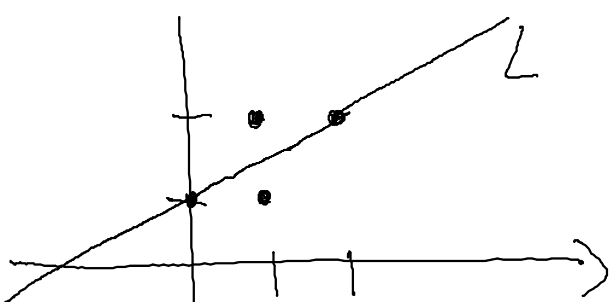



$$\left\| t^*\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}-2\\1\end{bmatrix} \right\| \le \left\| t\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}-2\\1\end{bmatrix} \right\| \quad \text{for all } t.$$

$$A = \begin{bmatrix}1\\1\end{bmatrix} \qquad b = \begin{bmatrix}-2\\1\end{bmatrix} \qquad x = t$$

$$\min_x \|Ax-b\| = \min_t \left\| \begin{bmatrix}1\\1\end{bmatrix} t - \begin{bmatrix}-2\\1\end{bmatrix} \right\|$$

**Example** Date $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}$




Find the straight line that best "fits" the data.

$y = ax + b \qquad a, b = ?$

$\begin{bmatrix}1\\1\end{bmatrix}$ in the line then $\Rightarrow 1 = a + b$ $\Big)$ $\begin{bmatrix}1 & 1\\ \end{bmatrix}\begin{bmatrix}a\\ \end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$

$\begin{bmatrix}1\\1\end{bmatrix}$ in the line then $\Rightarrow 1 = a+b$

$\begin{bmatrix}1\\2\end{bmatrix}$ " " " " $\Rightarrow 2 = a+b$

$\begin{bmatrix}2\\2\end{bmatrix}$ " " " " $\Rightarrow 2 = 2a+b$

$$\begin{bmatrix}1&1\\1&1\\2&1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}1\\2\\2\end{bmatrix}$$

Find $\begin{bmatrix}a\\b\end{bmatrix}$ that minimizes $\left\| \underbrace{\begin{bmatrix}1&1\\1&1\\2&1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}a\\b\end{bmatrix}}_{x}-\underbrace{\begin{bmatrix}1\\2\\2\end{bmatrix}}_{b}\right\|$

<u>Theorem</u>: The solutions $x$ to the least square problem $\boxed{Ax=b}$ (i.e. $x$ that minimizes $\|Ax-b\|$) are the solutions of $A^TAx = A^Tb$ (this system always has a solution)

<u>Back to the example</u>

$$\begin{bmatrix}1&1\\1&1\\2&1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}1\\2\\2\end{bmatrix}$$

$$\begin{bmatrix}1&1&2\\1&1&1\end{bmatrix}\begin{bmatrix}1&1\\1&1\\ & \end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}1&1&2\\1&1&1\end{bmatrix}\begin{bmatrix}1\\2\\2\end{bmatrix}$$

$$\begin{bmatrix}1&1&2\\1&1&1\end{bmatrix}\begin{bmatrix}1&1\\2&1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}1&1&1\end{bmatrix}\begin{bmatrix}2\end{bmatrix}$$

$$\begin{bmatrix}6&4\\4&3\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}7\\5\end{bmatrix}$$

$$\left[\begin{array}{cc|c}6&4&7\\4&3&5\end{array}\right] \quad \left[\begin{array}{cc|c}1&2/3&7/6\\0&1/3&1/3\end{array}\right] \quad \left[\begin{array}{cc|c}1&0&1/2\\0&1&1\end{array}\right]$$

Solution $\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}1/2\\1\end{bmatrix}$ $\quad y=\frac{x}{2}+1$

## Differential equations

**Def**: A differential equation is an equation where the unknown is a function, and some derivatives of the function appear in the equation. If the unknown is a function of only one variable, it is called an ordinary differential equation (ode)

**Example:** $y'' + 3xy = \cos x \quad (1)$

**Goal:** find $y = y(x)$

**Def:** The order of an ode is the order of the highest derivative in the equation

**Example:** The ode (1) is of order 2.

**Def:** An ode is said to be linear if it is of the form

$$a_n(x)\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \cdots + a_1(x)\, y' + a_0(x)\, y = g(x)$$

$a_0, a_1, \ldots, a_n, g$ are known functions of $x$.

**Notation:** $y^{(n)} = \dfrac{d^n y}{dx^n}$

**Example:** a) $3xy'' + \cos x \; y = e^x$ is linear

b) $(y')^2 + x^2 = 7\cos y$ is not linear (non-linear)

**Obs:** $y' = y \quad (2)$

$y = e^x$ and $y = 7e^x$ are both solutions of eq (2).

Most odes have many solutions

Example: $x' = 50$ Find $x = x(t)$

$x(0) = 5$

then $x(t) = 5 + 50t$. There is only one solution that satisfies both equations.

IVP. Initial value problem

$$y^{(n)} = F(x, y, y', \ldots, y^{(n-1)}) \quad \text{ode } (n^{th} \text{ order})$$

$$\left.\begin{aligned} y(x_0) &= y_0 \\ y'(x_0) &= y_1 \\ &\vdots \\ y^{(n-1)}(x_0) &= y_{n-1} \end{aligned}\right\} \text{initial values } (n \text{ of them})$$

In general, there will be one solution to these IVPs

Example: 1) First order IVP $\frac{dy}{dx} = f(x,y)$ $y(x_0) = y_0$

2) Second order IVP $y'' = f(x, y, y')$ $y(x_0) = y_0$

$y'(x_0) = y_1$

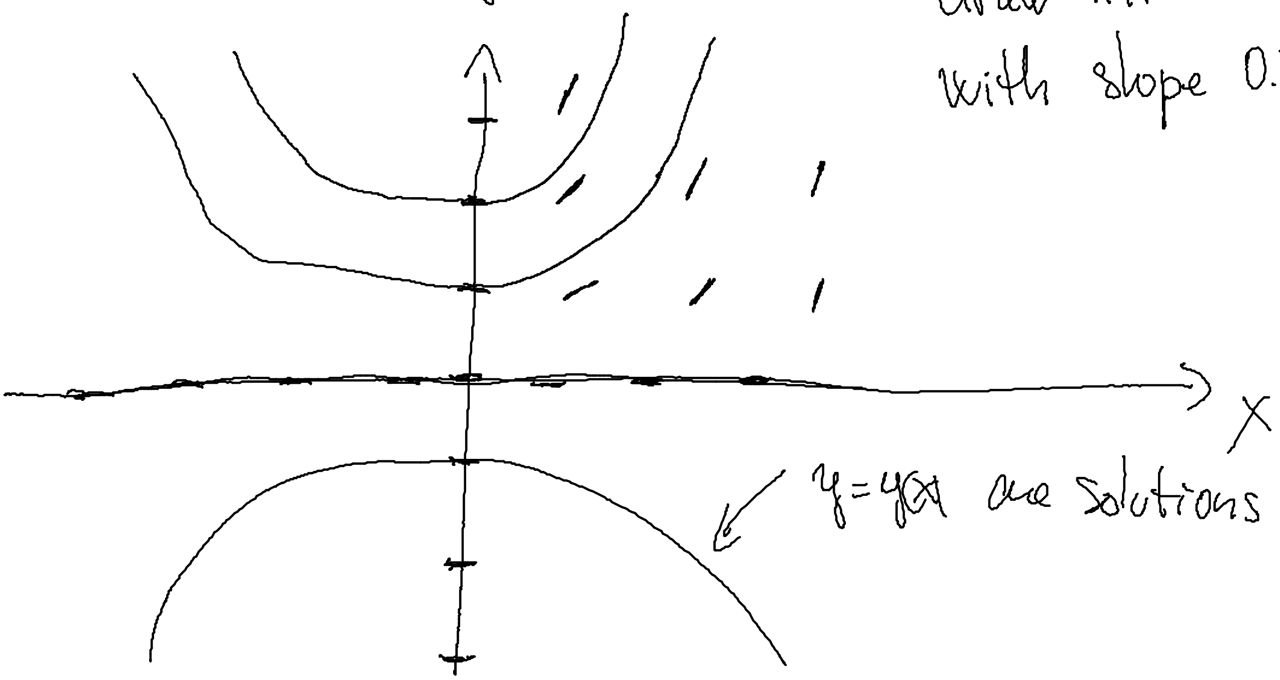
Understanding properties of the solutions without solving the ode

Direction fields: this is for first order equations

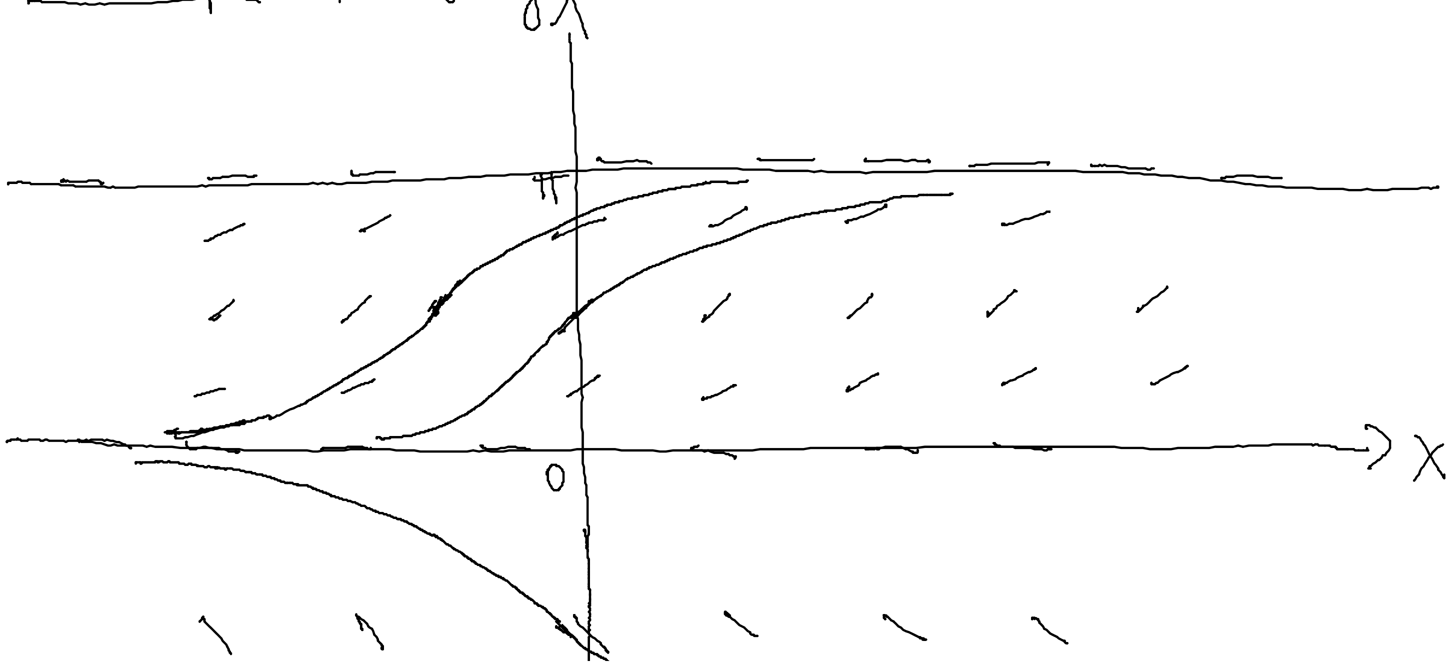
$$y' = f(x,y)$$

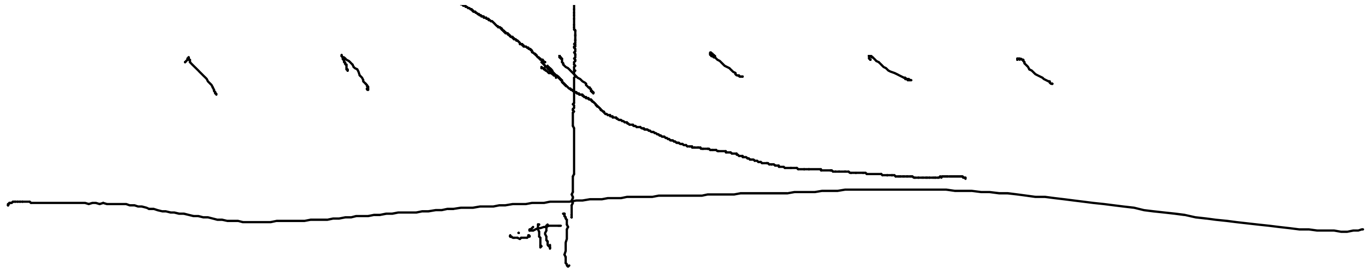
Example 1) $y' = 0.5\,x\,y$

In the $x,y$ plane, draw little segments with slope $0.5\,xy$




Example 2) $y' = \sin y$

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