$$\lim_{z \to z_0} f(z) = L$$

Examples: 1) $\lim_{z \to 1+i} z + 2z^2 = 1+i + 2(1+i)^2 = 1+5i$

2) $\lim_{z \to i} \frac{1}{i-z}$ does not exists

3) $\lim_{z \to i} \frac{z-i}{z^2+1} = \lim_{z \to i} \frac{\cancel{z-i}}{\cancel{(z-i)}(z+i)} = \frac{1}{2i} = -\frac{i}{2}$

Continuity: f is continuous at z_0 if $f(z_0) = \lim_{z \to z_0} f(z)$

Obs: 1) All polynomial are continuous

2) All rational functions are continuous except at the zeros of the denominator.

Derivatives: Let f be a function defined in a neighborhood of z_0. the derivative of f at z_0 is

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

if this limit exists.

Def: We say that f is differentiable at z_0 if $f'(z_0)$ exists.

Example: $f(z) = z^2$

$$\frac{f(z_0+h) - f(z_0)}{h} = \frac{(z_0+h)^2 - z_0^2}{h} = \frac{\cancel{z_0^2} + 2z_0h + h^2 - \cancel{z_0^2}}{h} =$$

$$\frac{f(z_0+h)-f(z_0)}{h} = \frac{(z_0+h)^2 - z_0^2}{h} = \frac{\cancel{z_0^2}+2z_0h+h^2-\cancel{z_0^2}}{h} =$$

$$= 2z_0 + h$$

$$\lim_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h} = 2z_0$$

$$f'(z) = \frac{d}{dz}(z^2) = \frac{df}{dz} = 2z$$

<u>Rules</u>: 1) $c \in \mathbb{C}$, then $\frac{dc}{dz} = 0$

2) $\frac{d}{dz}(f+g) = \frac{df}{dz} + \frac{dg}{dz}$

3) $\frac{d}{dz} fg = \frac{df}{dz}\, g + f\, \frac{dg}{dz}$

4) $\frac{d}{dz}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$

5) $\frac{d}{dz}\left[f(g(z))\right] = f'(g(z))\; g'(z)$

6) $\frac{d}{dz} z^n = n z^{n-1}$ for all integers n

<u>Example</u>

$f(z) = \mathrm{Re}(z)$ $\qquad z = x+iy$, $x, y \in \mathbb{R}$

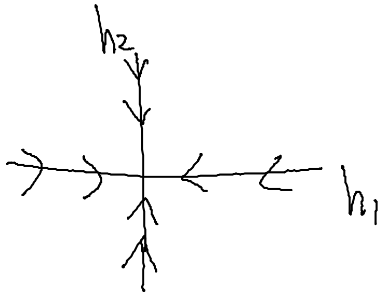
$f(z) = x$ $\qquad f(x,y) = (x,0)$ $\qquad h = h_1 + ih_2$

$h_1, h_2 \in \mathbb{R}$

$$\frac{f(z+h)-f(z)}{} = \frac{z_1+h_1-z_1}{} = \frac{h_1}{}$$

$h_1, h_2 \in \mathbb{R}$

$$\frac{f(z+h)-f(z)}{h} = \frac{z_1+h_1-z_1}{h_1+ih_2} = \frac{h_1}{h_1+ih_2}$$



$$\lim_{\substack{h_1\to 0\\ h_2=0}} \frac{h_1}{h_1+ih_2} = \lim_{h_1\to 0}\frac{h_1}{h_1} = 1$$

$\updownarrow \neq$

$$\lim_{\substack{h_1=0\\ h_2\to 0}} \frac{h_1}{h_1+ih_2} = \lim_{h_2\to 0}\frac{0}{ih_2} = 0$$

Then $\lim\limits_{h\to 0} \dfrac{h_1}{h_1+ih_2}$ does not exists.

Def: A complex valued function is said to be analytic at z_0 if it is differentiable in a neighborhood of z_0.

Cauchy-Riemman equations

$f(z) = u(x,y) + i\,v(x,y)$ $\quad z = x+iy \quad x, y, u \,\&\, v \in \mathbb{R}$

$h = h_1 + i h_2 \quad h_1, h_2 \in \mathbb{R}$

$$\boxed{f'(z)} = \lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \lim_{\substack{h_1\to 0\\ h_2=0}} \frac{[u(x+h_1,y)+i\,v(x+h_1,y)] - [u(x,y)+i\,v(x,y)]}{h_1}$$

$$= \lim_{h_1\to 0}\left[\frac{u(x+h_1,y)-u(x,y)}{h_1}\right] + i\left[\frac{v(x+h_1,y)-v(x,y)}{h_1}\right] = \boxed{\frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y)}$$

$$\frac{\partial x}{} \qquad \frac{}{\partial x}$$

$$\boxed{f'(z)} = \lim_{\substack{h_2\to 0\\ h_1=0}} \frac{f(z+h)-f(z)}{h} = \lim_{h_2\to 0}\frac{[u(x,y+h_2)+iv(x,y+h_2)] - [u(x,y)+iv(x,y)]}{ih_2} = \lim_{h_2\to 0}\left[\frac{u(x,y+h_2)-u(x,y)}{ih_2}\right] + i\left[\frac{v(x,y+h_2)-v(x,y)}{ih_2}\right] = \boxed{\frac{\partial v}{\partial y}(x,y) - i\frac{\partial u}{\partial y}(x,y)}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \qquad \boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

Cauchy-Riemman equations

f is differentiable if and only if $\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$ and in this case $f' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

↑ these are called the Cauchy-Riemman equations

Examples: 1) $f(z) = x \qquad u = x \qquad v = 0$

$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = 0$, the C-R equations are not satisfied, thus f is not differentiable.

2) $f(z) = z + 2z^2 \qquad \frac{\partial u}{\partial} = 1 + 4x \qquad \nearrow \frac{\partial v}{\partial x} = 4y$

2) $f(z) = z + 2z^2$

$u = x + 2x^2 - 2y^2$

$v = y + 4xy$

$\frac{\partial u}{\partial x} = 1 + 4x$ $\qquad$ $\frac{\partial v}{\partial x} = 4y$

$\frac{\partial u}{\partial y} = -4y$ $\qquad$ $\frac{\partial v}{\partial y} = 1 + 4x$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ✓ $\qquad$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ✓ thus,

$f(z) = z + 2z^2$ is differentiable, and

$$f' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 1 + 4x + i4y = 1 + 4(x+iy) = 1 + 4z$$

<u>Obs</u>: $f = u + iv$ analytic. C-R implies

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial}{\partial x}\left(\quad \downarrow \quad \right) \qquad \frac{\partial}{\partial y}\left(\quad \downarrow \quad \right)$$

$$(1) \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad (2) \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

$$(1) + (2) \quad \boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

Similarly, $\boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0}$

<u>Def</u>: $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ is called Laplace's equation

Def: $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ is called Laplace's equation

Obs: the real and imaginary part of analytic functions satisfy Laplace's equation.

Obs: Any real valued function of two variables that satisfies Laplace's equation is the real part of a complex valued analytic function. It is also the imaginary part of some other complex valued analytic function

Exponential: $z = x+iy$

$$f(z) = e^z = e^{x+iy} = e^x(\cos y + i \sin y)$$

$$u = e^x \cos y \qquad v = e^x \sin y$$

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial u}{\partial y} = -e^x \sin y$$

$$\parallel \qquad \checkmark \qquad \parallel \qquad \checkmark$$

$$\frac{\partial v}{\partial y} = e^x \cos y \qquad -\frac{\partial v}{\partial x} = -e^x \sin y$$

$f(z) = e^z$ is analytic everywhere.

Properties: 1) $e^{z_1+z_2} = e^{z_1} e^{z_2}$

$z_1 = x_1 + iy_1 \quad z_2 = x_2 + iy_2$

$z_1 = x_1 + iy_1 \qquad z_2 = x_2 + iy_2$

$$\boxed{e^{z_1} e^{z_2}} = e^{x_1}(\cos y_1 + i\sin y_1)\, e^{x_2}(\cos y_2 + i\sin y_2) =$$

$$= e^{x_1+x_2}\left[(\cos y_1 \cos y_2 - \sin y_1 \sin y_2) + i(\cos y_1 \sin y_2 + \sin y_1 \cos y_2)\right] = e^{x_1+x_2}\left[\cos(y_1+y_2) + i\sin(y_1+y_2)\right] =$$

$$e^{x_1+x_2}\, e^{i(y_1+y_2)} = e^{(x_1+x_2)+i(y_1+y_2)} = \boxed{e^{z_1+z_2}}$$

2) $$e^{z_1 - z_2} = \frac{e^{z_1}}{e^{z_2}}$$

3) $$\boxed{\frac{de^z}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = u + iv = e^z}$$

Logarithms: $w = \ln z$ if $e^w = z \qquad z \in \mathbb{C}, \; z \neq 0$

Goal: Find w as a function of z

$z = x + iy \qquad w = u + iv \qquad e^w = z$

$e^u(\cos v + i\sin v) = x + iy$

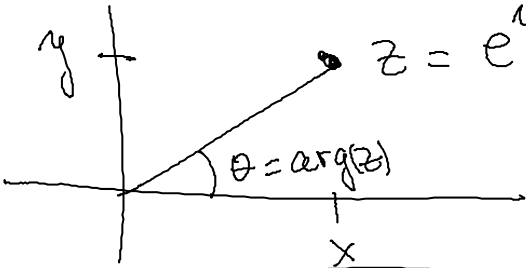
$e^u \cos v = x \qquad e^u \sin v = y$

square & add

$$e^{2u}\cos^2 v + e^{2u}\sin^2 v = x^2 + y^2$$

$e^{2u} = x^2+y^2 \qquad e^{u} = \sqrt{x^2+y^2} = |z|$

$\boxed{u = \ln|z|}$



$z = e^{u}(\cos v + i\sin v)$

$\boxed{v = \arg z + 2\pi k}$ k integer

$\boxed{\ln z = \ln|z| + i(\arg(z) + 2\pi k)}$ k integer

Obs: e^z is not one to one.

$e^{z_1} = e^{z_2}$ then $\dfrac{e^{z_1}}{e^{z_2}} = 1$ then $e^{z_1 - z_2} = 1$

$1 = |e^{z_1 - z_2}| = e^{Re(z_1 - z_2)}$ then $Re(z_1 - z_2) = 0$

then $z_2 - z_1 = 2\pi k i$ for some integer k.

Trigonometric functions

Obs: $y \in \mathbb{R}$, then $e^{iy} = \cos y + i\sin y$

$e^{-iy} = \cos y - i\sin y$

Add & devide by 2 $\quad \boxed{\cos y = \dfrac{e^{iy}+e^{-iy}}{2}}$

Substract & devide by 2i $\quad \boxed{\sin y = \dfrac{e^{iy}-e^{-iy}}{2i}}$

Def: $z \in \mathbb{C}$. $\boxed{\cos z = \dfrac{e^{iz}+e^{-iz}}{2}}$ and

$$\boxed{\sin z = \frac{e^{iz}-e^{-iz}}{2i}}$$