Trigonometric functions

Def: $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Obs: $\dfrac{d}{dz} \cos z = \dfrac{i}{2} e^{iz} - \dfrac{i}{2} e^{-iz} = (-1) \dfrac{e^{iz} - e^{-iz}}{2i} =$

$$= -\sin z$$

$$\frac{d}{dz} \sin z = \cos z$$

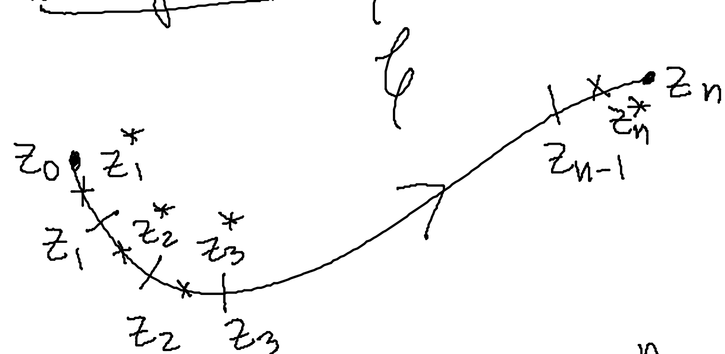
Obs: the trigonometric identities remain valid

Ex: $\cos^2 z + \sin^2 z = 1$

$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

Integrals: $\mathcal{C}$ is a curve in the complex plane

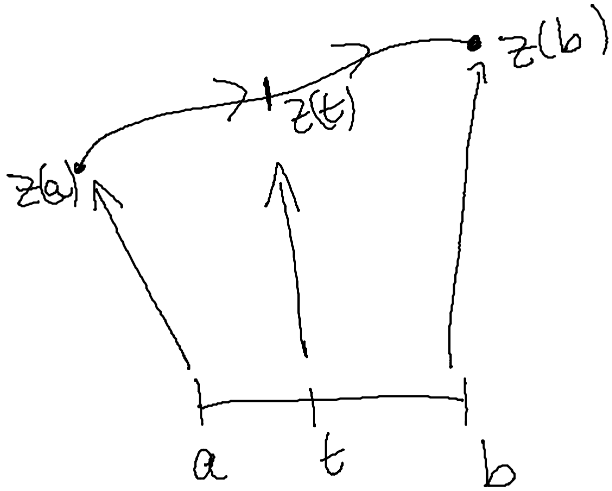


$$\int f(z)\,dz = \lim \sum^{n} f(z_k^*)(z_k - z_{k-1})$$

$$\int_{\mathcal{C}} f(z)\,dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(z_k^*)(z_k - z_{k-1})$$

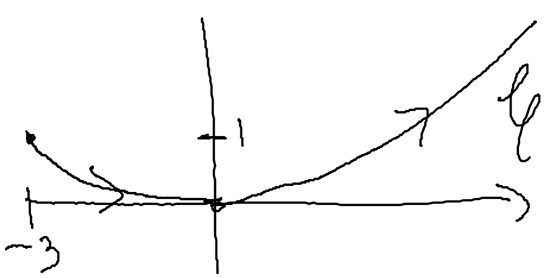
<u>Evaluation of integrals</u>

1) Parametrize $\mathcal{C}$. Find $z = z(t)$ a function defined on a real interval $z : [a,b] \to \mathcal{C}$. As t goes from a to b, you move in the direction of $\mathcal{C}$ and visit each point exactly once.



$$\int_{\mathcal{C}} f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

<u>Examples</u>: 1) $\mathcal{C}$ given by $x = 3t$; $y = t^2$; $-1 \le t \le 4$



Compute $\int_{\mathcal{C}} \bar{z}\,dz$

$f(z) = \bar{z}$

$$\int_{\mathcal{C}} \bar{z}\,dz = \int_{\mathcal{C}} f(z)\,dz = \int^b f(z(t))\,z'(t)\,dt = \int^4 \left(\overline{3t + it^2}\right)$$

$$\int_{\mathcal{C}} \bar{z}\, dz = \int_{\mathcal{C}} f(z)\, dz = \int_a f(z(t))\, z'(t)\, dt = \int_{-1} \overline{(3t+it^2)}\,(3+i2t)\, dt$$

$$\boxed{z(t) = 3t + it^2}$$

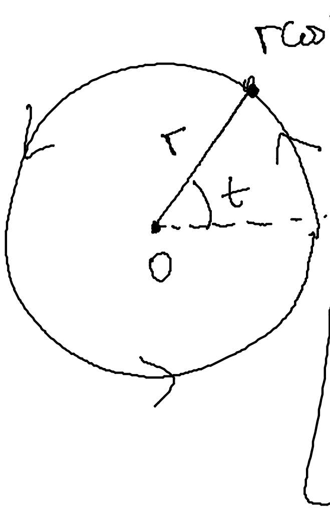
$$\int_{\mathcal{C}} \bar{z}\, dz = \int_{-1}^{4} \overline{3t+it^2}\,(3+i2t)\, dt = \int_{-1}^{4} (3t-it^2)(3+i2t)\, dt =$$

$$= \int_{-1}^{4} (9t+2t^3) + i(6t^2-3t^2)\, dt = \int_{-1}^{4} (9t+2t^3)\, dt +$$

$$+ i\int_{-1}^{4} 3t^2\, dt = \left(\frac{9}{2}t^2 + \frac{1}{2}t^4\right)\Big|_{-1}^{4} + i\; t^3\Big|_{-1}^{4} = \frac{9}{2}4^2 + \frac{4^4}{2} -$$

$$- \left(\frac{9}{2} + \frac{1}{2}\right) + i\left(4^3 + 1\right)$$

Example: $\mathcal{C}$ is the circle centered at 0 of radius r in the counter clockwise direction. Compute $\int_{\mathcal{C}} \frac{1}{z}\, dz$



$$z(t) = re^{it} = r(\cos t + i\sin t)$$

$$0 \le t \le 2\pi$$

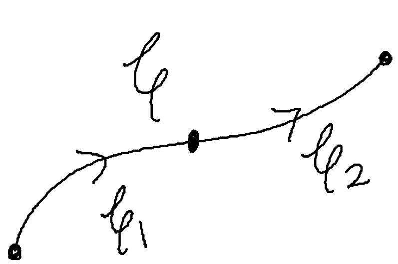
$$\boxed{\int_{\mathcal{C}} \frac{dz}{z} = \int_0^{2\pi} \frac{1}{\cancel{r}\cancel{e^{it}}}\,\cancel{r}\, i\,\cancel{e^{it}}\, dt} = \int_0^{2\pi} (0 + 1i)\, dt =$$

$$= \boxed{2\pi i}$$

$= 2\pi i$

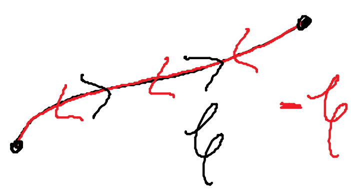
Properties: 1) $\int_{\mathcal{C}} c f(z)\,dz = c\int_{\mathcal{C}} f(z)\,dz$

2) $\int_{\mathcal{C}} (f(z)+g(z))\,dz = \int_{\mathcal{C}} f(z)\,dz + \int_{\mathcal{C}} g(z)\,dz$

3) 

$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$

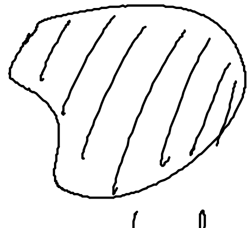
$$\int_{\mathcal{C}} f(z)\,dz = \int_{\mathcal{C}_1} f(z)\,dz + \int_{\mathcal{C}_2} f(z)\,dz$$

4) 

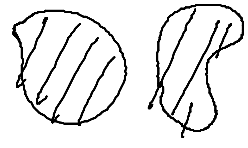
$-\mathcal{C}$ is the same curve as $\mathcal{C}$ but traveled in the opposite direction.

$$\int_{-\mathcal{C}} f(z)\,dz = -\int_{\mathcal{C}} f(z)\,dz$$

Def: 1) Connected sets

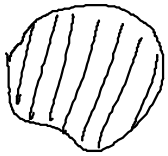


connected

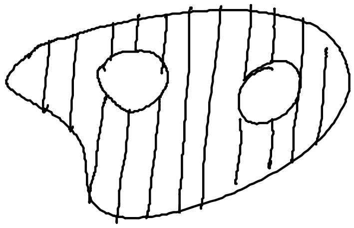


not connected

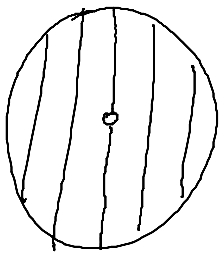
2) Simply connected: Connected plus it does not have holes



simply connected



not simply connected



$0 < |z| \leq 1$ not simply connected

Theorem: D open and simply connected. f analytic in D ($f'(z)$ exists for all $z \in D$). $\mathcal{C}$ a **closed** curve. Then

$$\int_{\mathcal{C}} f(z)\,dz = 0$$

Examples: 1) $\mathcal{C}$: circle of radius one centered at 0.

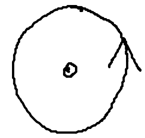
$f(z) = z \qquad z(t) = e^{it} \qquad 0 \leq t \leq 2\pi$

$$\int_{\mathcal{C}} z\,dz = \int_0^{2\pi} e^{it}\, i e^{it}\, dt = i \int_0^{2\pi} e^{2it}\, dt = \frac{i}{2i} e^{2it}\Big|_0^{2\pi}$$

$$\int_{\mathcal{C}} z\,dz = \int_0^{2\pi} e^{it}\, i e^{it}\, dt = i \int_0^{2\pi} e^{2it}\, dt = \frac{i}{2i} e^{2it}\Big|_0^{2\pi}$$

$$= \frac{1}{2}\Big(\underbrace{e^{2i2\pi}}_{=1} - \underbrace{e^{2i0}}_{=1}\Big) = 0$$

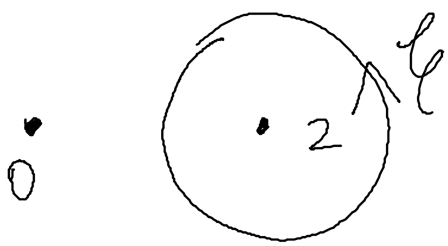
2) Same $\mathcal{C}$. $f(z) = \frac{1}{z}$



$$\int_{\mathcal{C}} \frac{1}{z}\, dz = 2\pi i \neq 0$$

O.K. because $\frac{1}{z}$ is not analytic at every point inside $\mathcal{C}$.

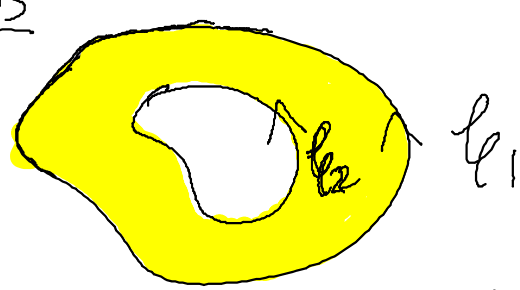
3)



$\mathcal{C} = \{|z-2| = 1\}$

$$\int_{\mathcal{C}} \frac{dz}{z} = 0$$

<u>Obs</u>



$\mathcal{C}_1$ & $\mathcal{C}_2$ closed curves.

$\mathcal{C}_2$ inside $\mathcal{C}_1$.

f analytic in $\mathcal{C}_1$, $\mathcal{C}_2$, and in the region between $\mathcal{C}_1$ and $\mathcal{C}_2$. Then

$$\int f(z)\,dz = \int f(z)\,dz$$

$\mathcal{C}_2$. Then $\displaystyle\int_{\mathcal{C}_1} f(z)\,dz = \int_{\mathcal{C}_2} f(z)\,dz$

Why?

$\mathcal{C}_1$ then $\mathcal{C}_3$ then $-\mathcal{C}_2$ then

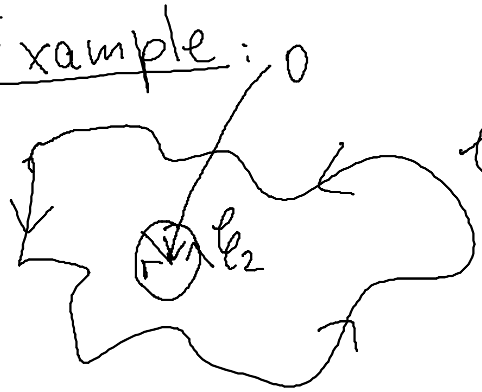
$\mathcal{C}_4 = -\mathcal{C}_3$

$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_3 \cup -\mathcal{C}_2 \cup -\mathcal{C}_3$$

$$\int_{\mathcal{C}} f(z)\,dz = 0 = \int_{\mathcal{C}_1} f(z)\,dz + \int_{\mathcal{C}_3} f(z)\,dz - \int_{\mathcal{C}_2} f(z)\,dz - \int_{\mathcal{C}_3} f(z)\,dz$$

then $\boxed{\displaystyle\int_{\mathcal{C}_1} f(z)\,dz = \int_{\mathcal{C}_2} f(z)\,dz}$

Example:



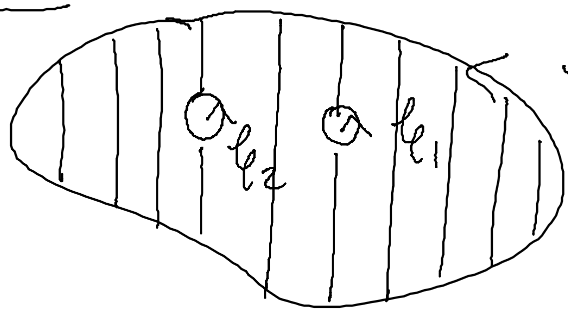
$$\int_{\mathcal{C}} \frac{1}{z}\,dz = \int_{\mathcal{C}_2} \frac{1}{z}\,dz = 2\pi i$$

Notation: $\mathcal{C}$ a closed curve, then $\displaystyle\oint_{\mathcal{C}} f(z)\,dz = \int_{\mathcal{C}} f(z)\,dz$ in the counterclockwise direction

Obs:

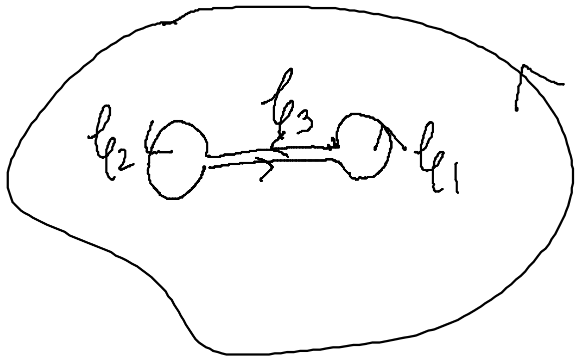
f analytic inside $\mathcal{C}$ but

Obs:



$\mathcal{C}$ f analytic inside $\mathcal{C}$ and also outside $\mathcal{C}_1$ & $\mathcal{C}_2$.

$$\oint_{\mathcal{C}} f(z)\,dz = \oint_{\mathcal{C}_1} f(z)\,dz + \oint_{\mathcal{C}_2} f(z)\,dz$$

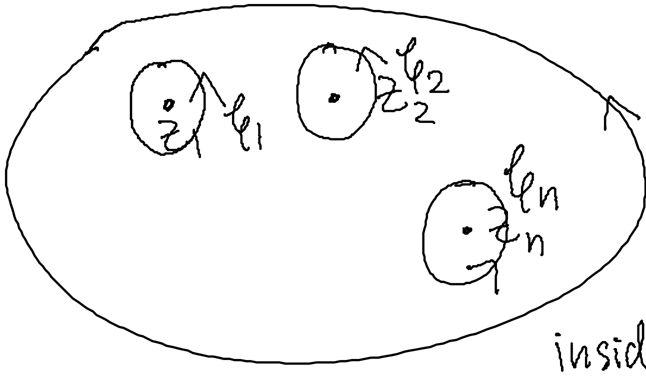


$\mathcal{C}$

$$\overline{\mathcal{C}} = \mathcal{C}_1 \cup \mathcal{C}_3 \cup \mathcal{C}_2 \cup -\mathcal{C}_3$$

$$\oint_{\mathcal{C}} = \oint_{\overline{\mathcal{C}}} = \oint_{\mathcal{C}_1} + \int_{\mathcal{C}_3} + \oint_{\mathcal{C}_2} - \int_{\mathcal{C}_3}$$

Obs $\mathcal{C}$ closed curve. f analytic inside $\mathcal{C}$ except at $z_1, z_2, \dots, z_n$



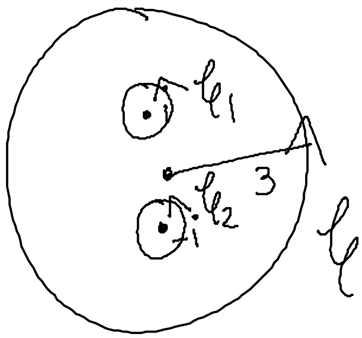
$\mathcal{C}$ $\mathcal{C}_i = \{ |z - z_i| = \varepsilon \}$

ε small so that the $\mathcal{C}_i$ are inside $\mathcal{C}$ & the $\mathcal{C}_i$ do not intersect.

then $$\oint_{\mathcal{C}} f(z)\,dz = \oint_{\mathcal{C}_1} f(z)\,dz + \dots + \oint_{\mathcal{C}_n} f(z)\,dz$$

$$\text{Ex: } \underset{|z|=3}{\overset{\mathcal{C}}{\oint}} \frac{dz}{z^2+1} = \underset{|z+i|=\varepsilon}{\overset{\mathcal{C}_1}{\oint}} \frac{dz}{z^2+1} + \underset{|z-i|=\varepsilon}{\overset{\mathcal{C}_n}{\oint}} \frac{dz}{z^2+1} = 0$$

because



$$\boxed{\oint_{|z+i|=\varepsilon} \frac{dz}{z^2+1}} = \int_0^{2\pi} \frac{\varepsilon i e^{it}\, dt}{-2i\varepsilon e^{it} + \varepsilon^2 e^{2it}} = \int_0^{2\pi} \frac{i\, dt}{-2i + \varepsilon e^{it}} \underset{\varepsilon \to 0}{=}$$

$$z = -i + \varepsilon e^{it}$$

$$= \int_0^{2\pi} \frac{i}{-2i}\, dt = \boxed{-\pi}$$

$$\boxed{\oint_{|z-i|=\varepsilon} \frac{dz}{z^2+1} = \pi}$$