Note Title 5/20/2015

Matrices

Def: A is a k by n matrix if A is of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{bmatrix} \quad \text{where } a_{ij} \in \mathbb{R}.$$

a_{ij} are the entries or components of A

We denote by $\mathbb{R}^{k \times n}$ the set of all k by n matrices.

Example $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 3 \end{bmatrix} \in \mathbb{R}^{2\times3}$

$a_{11} = 2 \quad a_{12} = -1 \quad a_{13} = 0$

$a_{21} = 1 \quad a_{22} = 1 \quad a_{23} = 3$

Vectors

Def: An n-vector is an $n \times 1$ matrix

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} \qquad x_i \in \mathbb{R}$$

Sometimes we write vectors as rows

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \end{bmatrix} = (x_1, x_2, \ldots, x_n) \qquad \mathbb{R}^n = \text{set of } n\text{-vectors}$$

$\begin{bmatrix} \vdots \\ x_n \end{bmatrix}$

Operations with matrices

Addition: A and $B \in \mathbb{R}^{k \times n}$

$C = A + B \qquad c_{ij} = a_{ij} + b_{ij}$

Example $\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -2 \\ 4 & 2 & 4 \end{bmatrix}$

Matrix-scalar multiplication

$\lambda \in \mathbb{R} \quad A \in \mathbb{R}^{k \times n}$ then $B = \lambda A \in \mathbb{R}^{k \times n}$

$b_{ij} = \lambda a_{ij}$ **Example:** $2 \begin{bmatrix} -1 & 0 & 1 \\ 2 & 5 & 7 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 2 \\ 4 & 10 & 14 \end{bmatrix}$

Obs: $A\lambda = \lambda A$

Properties $A, B, C \in \mathbb{R}^{k\times n}$ $\lambda, \beta \in \mathbb{R}$

1) $A+B = B+A$

2) $A+(B+C) = (A+B)+C$

3) $(\lambda\beta)A = \lambda(\beta A)$

4) $\lambda(A+B) = \lambda A + \lambda B$

5) $(\lambda+\beta)A = \lambda A + \beta A$

Matrix-vector multiplication

Def: $A \in \mathbb{R}^{k\times n}$ $x \in \mathbb{R}^n$. Let a_j be the j^{th} column of A, i.e.

$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ $a_j \in \mathbb{R}^k$. then

$$Ax = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

Example $\underbrace{\begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}}_{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} 2 + \begin{bmatrix} -1 \\ 0 \end{bmatrix}(-1) + \begin{bmatrix} 2 \\ -3 \end{bmatrix} 0 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Obs: $A \in \mathbb{R}^{k \times n}$ $\quad x \in \mathbb{R}^n$

$$Ax = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \\ \vdots \\ a_{k1} x_1 + a_{k2} x_2 + \cdots + a_{kn} x_n \end{bmatrix}$$

Example $\underbrace{\begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & -3 \end{bmatrix}}_{\in \mathbb{R}^{2\times 3}} \underbrace{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}}_{\in \mathbb{R}^3} = \begin{bmatrix} 1(2)+(-1)(-1)+2(0) \\ 2(2)+0(-1)+(-3)0 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}_{\in \mathbb{R}^2}$

Obs: $A \in \mathbb{R}^{k\times n}$ and $x \in \mathbb{R}^n$ then $Ax \in \mathbb{R}^k$

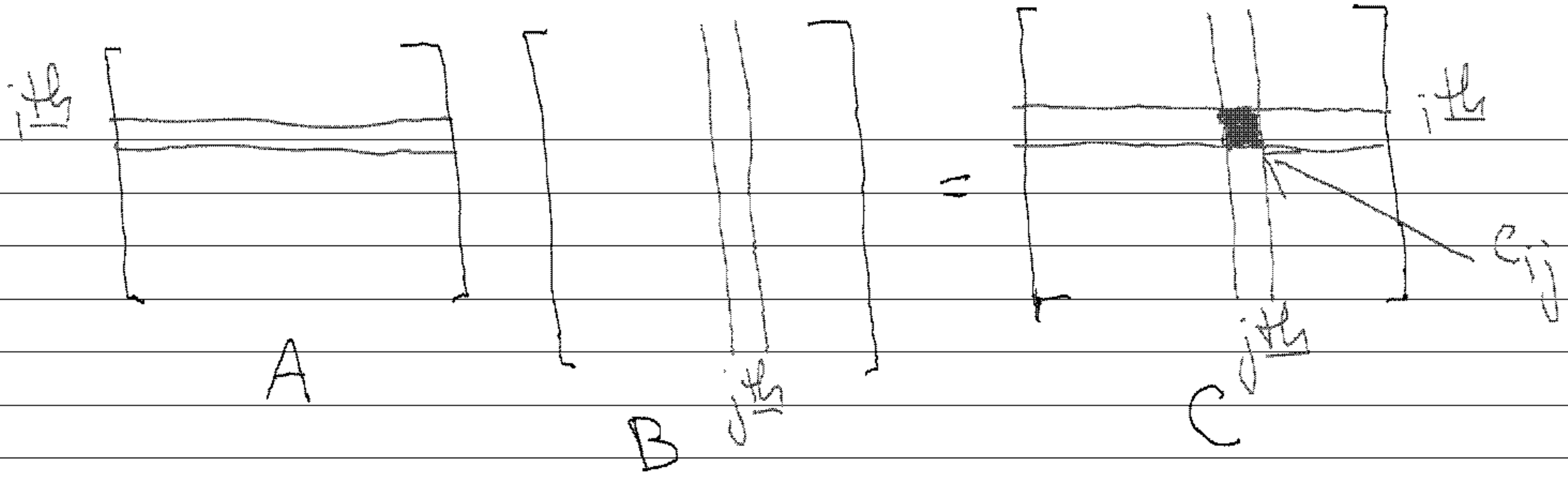
Matrix-matrix multiplication

Def: $A \in \mathbb{R}^{k\times p}$, $B \in \mathbb{R}^{p\times n}$ then $C = AB \in \mathbb{R}^{k\times n}$

Let c_j be the j^{th} column of C. Let b_j be the j^{th} column of B. Then $c_j = A b_j$.

Obs: $AB = A\begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix} = \begin{bmatrix} Ab_1 & Ab_2 & \dots & Ab_n \end{bmatrix}$

Obs:



$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{ip} b_{pj}$$

Example

$$\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2(1)+3(-1) & 2(0)+3(1) \\ (-1)1+1(-1) & -1(0)+1(1) \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -2 & 1 \end{bmatrix}$$

Properties

1) $A(B+C) = AB + AC$
2) $A(BC) = (AB)C$
3) $(A+B)C = AC + BC$

<u>Transpose of a matrix</u> $A \in \mathbb{R}^{k \times n}$. The transpose of A is $A^T \in \mathbb{R}^{n \times k}$. The j^{th} row of A^T is the j^{th} column of A

<u>Example</u> $\begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}^T = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}$

<u>Properties</u>: 1) $(A^T)^T = A$ 3) $(A+B)^T = A^T + B^T$

2) $(AB)^T = B^T A^T$ 4) $(\lambda A)^T = \lambda A^T$

<u>Special matrices</u>:

1) 0 is the matrix whose entries are all equal to zero

2) Upper triangular

$$\begin{bmatrix} x & \cdots & \cdots & x \\ 0 & x & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & x \end{bmatrix} \in \mathbb{R}^{n \times n}$$

← diagonal

3) lower triangular

$$\begin{bmatrix} x & 0 & \cdots & 0 \\ \vdots & x & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ x & \cdots & \cdots & x \end{bmatrix}$$

4) Diagonal

$$\begin{bmatrix} x & 0 & \cdots & 0 \\ 0 & x & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x \end{bmatrix}$$

5) Identity

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

6) Symmetric $A^T = A$

Obs: $AI = A$ and $IA = A$

Def: a linear equation is an equation of the form

$$\boxed{a_1 X_1 + a_2 X_2 + \cdots + a_n X_n = b}$$

where $a_1, a_2, \ldots, a_n, b$ are given numbers, and $X_1, X_2, \ldots, X_n$ are the unknowns or variables

Def: System of linear equations

$$a_{11} X_1 + a_{12} X_2 + \cdots + a_{1n} X_n = b_1$$
$$a_{21} X_1 + a_{22} X_2 + \cdots + a_{2n} X_n = b_2$$

$$\boxed{\begin{array}{c} \vdots \qquad\qquad \vdots \\ a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n = b_k \end{array}}$$

a_{ij} and b_i are known numbers

<u>Goal</u>: Find all $x_1, x_2, \ldots, x_n$ that satisfy the equations

In matrix-vector notation

$A \in \mathbb{R}^{k \times n}$ $\quad b \in \mathbb{R}^k$ both given

Find all $x \in \mathbb{R}^n$ such that $\boxed{Ax = b}$

<u>Augmented matrix of a linear system</u>

<u>Def</u>: The augmented matrix of $Ax = b$ is $[A \mid b]$

Example: $2x_1 - x_2 + x_3 = 5$
$3x_2 + 2x_3 = -2$

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

Augmented matrix $\left[\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 0 & 3 & 2 & -2 \end{array}\right]$

Def: A system is said to be consistent if it has at least one solution. Otherwise, it is inconsistent

Obs: A system may have:
1) no solutions
2) only one solution
3) An infinite number of solutions

Def: A matrix is said to be reduced row echelon (RRE) if it is of the form

$$\begin{bmatrix} \boxed{1} & 0 & x & 0 & x & x & x \\ 0 & \boxed{1} & x & 0 & x & x & x \\ 0 & 0 & 0 & \boxed{1} & x & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{1}$ are called leading ones. The entries of the leading ones are called pivots. The columns that contain the pivot entries are called pivot columns

1) The first nonzero entry in a row is a one, called leading one
2) If a row does not have a leading one, neither does the next row

3) the leading one in a row is to the right of the leading one in the previous row

4) the only non-zero element in a pivot column is the leading one.

<u>Solving $Ax=b$ when $[A|b]$ is RRE</u>

<u>Case 1</u>: $[A|b]$ has a leading one in the last column

$$[A|b] = \begin{bmatrix} 0 \cdots\cdots 0 & \boxed{1} \end{bmatrix}$$

the system is inconsistent (has no solutions)

Case 2: the last column of $[A|b]$ is not a pivot column

Def: x_i is a leading variable if the i^{th} column of $[A|b]$ is a pivot column. Otherwise x_i is a free variable.

Label each free variable with a parameter, $t_1, t_2, \ldots, t_r$ if we have r free variables.

Move all the free variables to the right hand side

the parameters $t_1, t_2, \ldots, t_r$ can take any values and that gives you all the solutions

Example

$$[A \mid b] = \left[\begin{array}{ccccc|c} \boxed{1} & -1 & 0 & 2 & 0 & -1 \\ 0 & 0 & \boxed{1} & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & \boxed{1} & 7 \end{array}\right]$$

leading variables: x_1, x_3, x_5

free variables: x_2, x_4

$x_1 = -1 + t_1 - 2t_2$ $\quad t_1, t_2 \in \mathbb{R}$

$x_2 = t_1$

$x_3 = 1 + 2t_2$

$x_4 = t_2$

$x_5 = 7$

$$x = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 7 \end{bmatrix} + t_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$