Lecture Notes 4

3.2 Ratio of areas

In the previous subsection we gave a geometric interpretation for the sign of Gaussian curvature. Here we describe the geometric significance of the magnitude of $K$.

If $V$ is a sufficiently small neighborhood of $p$ in $M$ (where $M$, as always, denotes a regular embedded surface in $\mathbb{R}^3$), then it is easy to show that there exist a patch $(U, X)$ centered at $p$ such that $X(U) = V$. Area of $V$ is then defined as follows:

$$\text{Area}(V) := \int \int_D \|D_1 X \times D_2 X\| \, du^3 du^2.$$  

Using the chain rule, one can show that the above definition is independent of the patch.

**Exercise 3.2.1.** Let $V \subset S^2$ be a region bounded in between a pair of great circles meeting each other at an angle of $\alpha$. Show that $\text{Area}(V) = 2\alpha$ (*Hints: Let $U := [0, \alpha] \times [0, \pi]$ and $X(\theta, \phi) := (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi)$. Show that $\|D_1 X \times D_2 X\| = |\sin \phi|$. Further, note that, after a rotation we may assume that $X(U) = V$. Then an integration will yield the desired result*).

**Exercise 3.2.2.** Use the previous exercise to show that the area of a geodesic triangle $T \subset S^2$ (a region bounded by three great circles) is equal to sum of its angles minus $\pi$ (*Hints: Use the picture below: $A + B + C + T = 2\pi$, and $A = 2\alpha - T$, $B = 2\beta - T$, and $C = 2\gamma - T$).

Let $V_r := B_r(p) \cap M$. Then, if $r$ is sufficiently small, $V(r) \subset X(U)$, and, consequently, $U_r := X^{-1}(V_r)$ is well defined. In particular, we may compute the area of $V_r$ using the patch $(U_r, X)$. In this section we show that

$$|K(p)| = \lim_{r \to 0} \frac{\text{Area}(n(V_r))}{\text{Area}(V_r)}.$$  

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Exercise 3.2.3. Recall that the mean value theorem states that $\int \int_U f \, du^1 du^2 = f(\tilde{\alpha}^1, \tilde{\alpha}^2) \text{Area}(U)$, for some $(\tilde{\alpha}^1, \tilde{\alpha}^2) \in U$. Use this theorem to show that

$$\lim_{r \to 0} \frac{\text{Area}(n(V_r))}{\text{Area}(V_r)} = \frac{\|D_1 N(0,0) \times D_2 N(0,0)\|}{\|D_1 X(0,0) \times D_2 X(0,0)\|}$$

(Recall that $N := n \circ X$.)

Exercise 3.2.4. Prove Lagrange's identity: for every pair of vectors $v, w \in \mathbb{R}^3$,

$$\|v \times w\|^2 = \det \begin{vmatrix} \langle v, v \rangle & \langle v, w \rangle \\ \langle w, v \rangle & \langle w, w \rangle \end{vmatrix}.$$

Now set $g(u^1, u^2) := \det[g_{ij}(u^1, u^2)]$. Then, by the previous exercise it follows that $\|D_1 X(0,0) \times D_2 X(0,0)\| = \sqrt{g(0,0)}$. Hence, to complete the proof of the main result of this section it remains to show that

$$\|D_1 N(0,0) \times D_2 N(0,0)\| = K(p)\sqrt{g(0,0)}.$$

We prove the above formula using two different methods:

Method 1. Recall that $K(p) := \det(S_p)$, where $S_p := -dn_p: T_p M \to T_p M$ is the shape operator of $M$ at $p$. Also recall that $D_i X(0,0), i = 1, 2$, form a basis for $T_p M$. Let $S_{ij}$ be the coefficients of the matrix representation of $S_p$ with respect to this basis, then

$$S_p(D_i X) = \sum_{j=1}^2 S_{ij} D_j X.$$

Further, recall that $N := n \circ X$. Thus the chain rule yields:

$$S_p(D_i X) = -dn_p(D_i X) = -D_i (n \circ X) = -D_i N.$$
Exercise 3.2.5. Verify the middle step in the above formula, i.e., show that $dn(D_i X) = D_i (n \circ X)$.

From the previous two lines of formulas, it now follows that

$$-D_i N = \sum_{j=1}^{2} S_{ij} D_j X.$$ 

Taking the inner product of both sides with $D_k N, k = 1, 2$, we get

$$\langle -D_i N, D_k N \rangle = \sum_{j=1}^{2} S_{ij} \langle D_j X, D_k N \rangle.$$ 

Exercise 3.2.6. Let $F, G: U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a pair of mappings such that $\langle F, G \rangle = 0$. Prove that $\langle D_i F, G \rangle = -\langle F, D_i G \rangle$.

Now recall that $\langle D_j X, N \rangle = 0$. Hence the previous exercise yields:

$$\langle D_j X, D_k N \rangle = -\langle D_{kj} X, N \rangle = -l_{ij}.$$ 

Combining the previous two lines of formulas, we get: $\langle D_i N, D_k N \rangle = \sum_{k=1}^{2} S_{ij} l_{jk}$; which in matrix notation is equivalent to

$$[\langle D_i N, D_j N \rangle] = [S_{ij}] [l_{ij}].$$ 

Finally, recall that $\det(\langle D_i N, D_j N \rangle) = \|D_1 N \times D_2 N\|^2$, $\det(S_{ij}) = K$, and $\det(l_{ij}) = K g$. Hence taking the determinant of both sides in the above equation, and then taking the square root yields the desired result.

Next, we discuss the second method for proving that $\|D_1 N \times D_2 N\| = K \sqrt{g}$.

METHOD 2. Here we work with a special patch which makes the computations easier:

Exercise 3.2.7. Show that there exist a patch $(U, X)$ centered at $p$ such that $[g_{ij}(0,0)]$ is the identity matrix. (Hint: Start with a Monge patch with respect to $T_p M$)

Thus, if we are working with the coordinate patch referred to in the above exercise, $g(0,0) = 1$, and, consequently, all we need is to prove that $\|D_1 N(0,0) \times D_2 N(0,0)\| = K(p)$.
Exercise 3.2.8. Let $f : U \subset \mathbb{R}^2 \to \mathbb{S}^2$ be a differentiable mapping. Show that $\langle D_i f(u^1, u^2), f(u^1, u^2) \rangle = 0$ (Hints: note that $\langle f, f \rangle = 1$ and differentiate).

It follows from the previous exercise that $\langle D_i N, N \rangle = 0$. Now recall that $N(0, 0) = n \circ X(0, 0) = n(p)$. Hence, we may conclude that $N(0, 0) \in T_p M$. Further recall that $\{D_1 X(0, 0), D_2 X(0, 0)\}$ is now an orthonormal basis for $T_p M$ (because we have chosen $(U, X)$ so that $[g_{ij}(0, 0)]$ is the identity matrix). Consequently,

$$D_i N = \sum_{k=1}^{2} \langle D_i N, D_k X \rangle D_k X,$$

where we have omitted the explicit reference to the point $(0, 0)$ in the above formula in order to make the notation less cumbersome (it is important to keep in mind, however, that the above is valid only at $(0, 0)$). Taking the inner product of both sides of this equation with $D_j N(0, 0)$ yields:

$$\langle D_i N, D_j N \rangle = \sum_{k=1}^{2} \langle D_i N, D_k X \rangle \langle D_k X, D_j N \rangle.$$

Now recall that $\langle D_i N, D_k X \rangle = -\langle N, D_{ij} X \rangle = -l_{ij}$. Similarly, $\langle D_k X, D_j N \rangle = -l_{kj}$. Thus, in matrix notation, the above formula is equivalent to the following:

$$[\langle D_i N, D_j N \rangle] = [l_{ij}]^2$$

Finally, recall that $K(p) = \det[l_{ij}(0, 0)]/\det[g_{ij}(0, 0)] = \det[l_{ij}(0, 0)]$. Hence, taking the determinant of both sides of the above equation yields the desired result.

3.3 Product of principal curvatures

For every $v \in T_p M$ with $\|v\| = 1$ we define the normal curvature of $M$ at $p$ in the direction of $v$ by

$$k_v(p) := \langle \gamma''(0), n(p) \rangle,$$

where $\gamma : (-\epsilon, \epsilon) \to M$ is a curve with $\gamma(0) = p$ and $\gamma'(0) = v$.

Exercise 3.3.1. Show that $k_v(p)$ does not depend on $\gamma$. 

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In particular, by the above exercise, we may take $\gamma$ to be a curve which lies in the intersection of $M$ with a plane which passes through $p$ and is normal to $n(p) \times v$. So, intuitively, $k_v(p)$ is a measure of the curvature of an orthogonal cross section of $M$ at $p$.

Let $UT_pM := \{v \in T_pM \mid \|v\| = 1\}$ denote the unit tangent space of $M$ at $p$. The principal curvatures of $M$ at $p$ are defined as

$$k_1(p) := \min_v k_v(p), \quad \text{and} \quad k_2(p) := \max_v k_v(p),$$

where $v$ ranges over $UT_pM$. Our main aim in this subsection is to show that

$$K(p) = k_1(p)k_2(p).$$

Since $K(p)$ is the determinant of the shape operator $S_p$, to prove the above it suffices to show that $k_1(p)$ and $k_2(p)$ are the eigenvalues of $S_p$.

First, we need to define the second fundamental form of $M$ at $p$. This is a bilinear map $\Pi_p: T_pM \times T_pM \to \mathbb{R}$ defined by

$$\Pi_p(v, w) := \langle S_p(v), w \rangle.$$

We claim that, for all $v \in UT_pM$,

$$k_v(p) = \Pi_p(v, v).$$

The above follows from the following computation

$$\langle S_p(v), v \rangle = -\langle dn_p(v), v \rangle = -\langle (n \circ \gamma)'(0), \gamma'(0) \rangle = \langle (n \circ \gamma)(0), \gamma''(0) \rangle = \langle n(p), \gamma''(0) \rangle$$

**Exercise 3.3.2.** Verify the passage from the second to the third line in the above computation, i.e., show that $-\langle (n \circ \gamma)'(0), \gamma'(0) \rangle = \langle (n \circ \gamma)(0), \gamma''(0) \rangle$ (Hint: Set $f(t) := \langle n(\gamma(t)), \gamma'(t) \rangle$, note that $f(t) = 0$, and differentiate.)

So we conclude that $k_v(p)$ are the minimum and maximum of $\Pi_p(v)$ over $UT_pM$. Hence, all we need is to show that the extrema of $\Pi_p$ over $UT_pM$ coincide with the eigenvalues of $S_p$.

**Exercise 3.3.3.** Show that $\Pi_p$ is symmetric, i.e., $\Pi_p(v, w) = \Pi_p(w, v)$ for all $v, w \in T_pM$. 

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By the above exercise, $S_p$ is a self-adjoint operator, i.e., $\langle S_p(v), w \rangle = \langle v, S_p(w) \rangle$. Hence $S_p$ is orthogonally diagonalizable, i.e., there exist orthonormal vectors $e_i \in T_pM, i = 1, 2$, such that

$$S_p(e_i) = \lambda_i e_i.$$ 

By convention, we suppose that $\lambda_1 \leq \lambda_2$. Now note that each $v \in UT_pM$ may be represented uniquely as $v = v^1e_1 + v^2e_2$ where $(v^1)^2 + (v^2)^2 = 1$. So for each $v \in UT_pM$ there exists a unique angle $\theta \in [0, 2\pi)$ such that

$$v(\theta) := \cos \theta e_1 + \sin \theta e_2;$$

Consequently, bilinearity of $\Pi_p$ yields

$$\Pi_p(v(\theta), v(\theta)) = \lambda_1 \cos^2 \theta + \lambda_2 \sin^2 \theta.$$

**Exercise 3.3.4.** Verify the above claim, and show that minimum and maximum values of $\Pi_p$ are $\lambda_1$ and $\lambda_2$ respectively. Thus $k_1(p) = \lambda_1$, and $k_2(p) = \lambda_2$.

The previous exercise completes the proof that $K(p) = k_1(p)k_2(p)$, and also yields the following formula which was discovered by Euler:

$$k_v(p) = k_1(p)\cos^2 \theta + k_2(p)\sin^2 \theta.$$ 

In particular, note that by the above formula there exists always a pair of orthogonal directions where $k_v(p)$ achieves its maximum and minimum values. These are known as the principal directions of $M$ at $p$. 
