Math 142 Sep 21, 2001
Calculus II
Fall 2001, USC

Solutions to Midterm 1

1a. $\ln e^{2x} = 2x$. So

$$y' = \Big(\tan(2x)\Big)' = \Big(\sec^2(2x)\Big)(2x)' = 2\sec^2(2x).$$

1b. Differentiating any function of the form $y = f(x)^{g(x)}$, such as $y = x^{\sin x}$, involves 3 steps:

(Step 1) Take natural log of both sides:

$$\begin{aligned} \ln y &= \ln(x^{\sin x}) \\ &= \sin x \cdot \ln x \end{aligned}$$

(Step 2) Differentiate both sides:

$$\begin{aligned} \frac{y'}{y} &= (\sin x)'(\ln x) + (\sin x)(\ln x)' \\ &= \cos x \cdot \ln x + \sin x \cdot \frac{1}{x}. \end{aligned}$$

(Step 3) Solve for y' in terms of x:

$$y' = x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right).$$

2a. *METHOD 1.* Let $y = a^x$. Then $\ln y = x \ln a$. So

$$\frac{y'}{y} = \ln a;$$
$$y' = y \ln a = a^x \ln a.$$

METHOD 2.

$$a^x = e^{\ln a^x} = e^{x \ln a}$$
$$(a^x)' = (e^{x\ln a})' = e^{x \ln a} \cdot (x \ln a)' = a^x \ln a.$$

2b. To differentiate an inverse trigonometric function, such as $\cos^{-1} x$, follow these 3 steps:

(Step 1) Let $y = \cos^{-1} x$. Then

$$\cos y = x.$$

(Step 2) Differentiate both sides:

$$(-\sin y)\, y' = 1.$$

(Step 3) Solve for y' in terms of x

$$y' = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1-\cos^2 y}} = \frac{-1}{\sqrt{1-x^2}}.$$

We put $\sin y = \sqrt{1-\cos^2 x}$ (as opposed to $-\sqrt{1-\cos^2 x}$) because by definition $0 \le y \le \pi$, which forces $\sin y \ge 0$.

3a. Let $u = x^2 + 1$. Then $du = 2xdx$, or $xdx = du/2$. So

$$\int \frac{x}{x^2+1}\, dx = \frac{1}{2} \int \frac{1}{u}\, du = \frac{1}{2} \ln u + C = \ln \sqrt{x^2+1} + C.$$

3b.

$$\begin{aligned}
\int \frac{1}{x^2+4x+8}\, dx &= \int \frac{1}{(x+2)^2+4}\, dx \\
&= \int \frac{1}{(u)^2+2^2}\, du \qquad (u = x+2) \\
&= \frac{1}{2} \tan^{-1} \frac{u}{2} + C \\
&= \frac{1}{2} \tan^{-1} \frac{x+2}{2} + C.
\end{aligned}$$

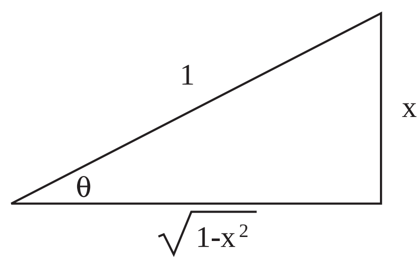
4a.

$$\cosh x - \sinh x = \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} = \frac{2\,e^{-x}}{2} = e^{-x}.$$

4b. *METHOD 1.* Let $y = \sin^{-1} x$. Then $\sin y = x$. Further, $\pi/2 \le y \le -\pi/2$. So $\cos y \ge 0$, which yields $\cos y = \sqrt{1 - \sin^2 x}$. Hence

$$\tan(\sin^{-1} x) = \tan y = \frac{\sin y}{\cos y} = \frac{x}{\sqrt{1 - x^2}}.$$

METHOD 2. Draw a triangle with $\sin \theta = x$.



Then

$$\tan(\sin^{-1} x) = \tan \theta = \frac{x}{\sqrt{1 - x^2}}.$$

5. Let $y(t)$ be the population at time t. By assumption, $y(0) = 1000$. So

$$y(t) = y(0)e^{kt} = 1000e^{kt}.$$

Further, we are given that $y(3) = 8000$. So $8000 = 1000e^{k\,3}$, which implies

$$k = \frac{1}{3} \ln \frac{8000}{1000} = \ln \sqrt[3]{8} = \ln 2.$$

Thus

$$y(5) = 1000e^{(\ln 2)5} = 1000e^{\ln 2^5} = 1000 \cdot 2^5 = 32000.$$