Solutions to Midterm 2

1a. To differentiate $y = \cos x^\sin x$, first take natural log of both sides:

$$\ln y = \sin x \ln(\cos x).$$

Then differentiate both sides:

$$\frac{y'}{y} = \sin' x \ln(\cos x) + \sin x(\ln(\cos x))'$$

$$= \cos x \ln(\cos x) + \sin x \left(\frac{1}{\cos x}(-\sin x)\right).$$

Finally, multiply both sides by $y$:

$$y' = \cos x^\sin x(\cos x \ln(\cos x) - \sin x \tan x).$$

1b. See the solution set to Midterm 1.

2a. To find $\int \ln x \, dx$, let

$$u = \ln x \quad \text{and} \quad dv = dx.$$ 

Then

$$du = \frac{1}{x} \, dx \quad \text{and} \quad v = x.$$ 

So integration by parts yields:

$$\int \ln x \, dx = x \ln x - \int \frac{x}{x} \, dx$$

$$= x \ln x - x + C$$

2b. To find $\int \sin^n x \, dx$, where $n$ is an odd integer, write the integrand as $\sin^{n-1} x \sin x$, and use the formula $\sin^2 x + \cos^2 x = 1$:

$$\int \sin^3 x \, dx = \int \sin^2 x \sin x \, dx = \int (1 - \cos^2 x) \sin x \, dx$$
Let \( u = \cos x \), then \( du = -\sin x \, dx \). So the above integral becomes:

\[
\int (1 - u^2)(-du) = -u + \frac{1}{3}u^3 + C = -\cos x + \frac{1}{3}\cos^3 x + C.
\]

2d.

\[
\frac{x - 7}{x^2 - x - 12} = \frac{x - 7}{(x - 4)(x + 3)} = \frac{A}{x - 4} + \frac{B}{x + 3} = \frac{A(x + 3) + B(x - 4)}{(x - 4)(x + 3)}
\]

So it follows that

\[
x - 7 = A(x + 3) + B(x - 4).
\]

Setting \( x = 4 \) on both sides of the above equation, we get

\[
4 - 7 = A(4 + 3) + B(4 - 4).
\]

So \(-3 = 7A\), which yields \( A = -3/7\). Similarly, setting \( x = -3 \), we get

\[
-3 - 7 = A(-3 + 3) + B(-3 - 4),
\]

which yields that \(-10 = -7B\), or \( B = 10/7 \). So

\[
\int \frac{x - 7}{x^2 - x - 12} \, dx = \int \frac{-3/7}{x - 4} \, dx + \int \frac{10/7}{x + 3} \, dx
\]

\[
= -\frac{3}{7}\ln(x - 4) + \frac{10}{7}\ln(x + 3) + C.
\]

2c. Let \( u = 1 - x \), then \( du = -dx \). So

\[
\int_0^1 \frac{dx}{\sqrt{1 - x}} = \int_{1-0}^{1-1} u^{-1/2}(-du)
\]

\[
= -\int_1^0 u^{-1/2} \, du
\]

\[
= \left. \frac{-1}{-1/2 + 1}u^{-1/2+1} \right|_1^0 = 2.
\]
3a. \( \lim_{x \to 0} (\cos x)^{\frac{1}{x}} \) is indeterminate of the form \( 1^\infty \). So proceed as follows:

\[
y = \cos x^{\frac{1}{x}} \\
\ln y = \frac{1}{x} \ln \cos x \\
\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{1}{x} \ln \cos x.
\]

Since the last limit above is indeterminate of the form \( 0/0 \), we may apply the L'Hopital’s rule:

\[
\lim_{x \to 0} \frac{\ln \cos x}{x} = \lim_{x \to 0} \frac{-\sin x / \cos x}{1} = \frac{0}{1} = 0.
\]

So \( \lim_{x \to 0} \ln y = 0 \). But, since \( \ln \) is continuous, \( \lim_{x \to 0} \ln y = \ln(\lim_{x \to 0} y) \).

So \( \ln(\lim_{x \to 0} y) = 0 \), which yields that

\[
\lim_{x \to 0} y = e^0 = 1.
\]

3b. \( \lim_{x \to 0} (x \ln x^2) \) is indeterminate of the form \( 0 \cdot \infty \), so we proceed as follows:

\[
\lim_{x \to 0} (x \ln x^2) = \lim_{x \to 0} \frac{\ln x^2}{1/x}.
\]

Now the limit on the right is of the form \( \infty /\infty \), so we may apply the L'Hopital’s rule:

\[
\lim_{x \to 0} \frac{\ln x^2}{1/x} = \lim_{x \to 0} \frac{2x/x^2}{-1/x^2} = \lim_{x \to 0} -2x = 0.
\]

So we conclude that \( a_n \) converges.

4a. The general term for the series \( \frac{1}{4}, \frac{2}{8}, \frac{3}{16}, \frac{4}{32}, \frac{5}{64}, \ldots \) is given by

\[
a_n = (-1)^n \frac{n}{2^{n+1}}.
\]

Since \( -|a_n| \leq a_n \leq |a_n| \), then \( a_n \) converges if and only if \( |a_n| \) converges.

The latter limit may be computed using the L’Hopital’s rule:

\[
\lim_{n \to \infty} |a_n| = \lim_{n \to \infty} \frac{n}{2^{n+1}} = \lim_{n \to \infty} \frac{1}{2^{n+1} \ln 2} = \frac{1}{\infty} = 0.
\]

So we conclude that \( a_n \) converges.
4b. Consider the following picture: Since the area under the graph of

\[ y = \frac{1}{x}, \text{ from } 1 \text{ to } n, \text{ is less than the sum of the areas of the first } n - 1 \text{ rectangles, we have} \]

\[
\ln n = \int_1^n \frac{1}{x} \, dx \leq 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1}.
\]

Since \( \lim_{n \to \infty} \ln n = \infty \), it follows that the series on the right hand side of the above inequality diverges.