Math 142
Calculus II
Fall 2001, USC

Nov 16, 2001

Solutions To Midterm 3

1a. This is best done using *logarithmic differentiation*. First take natural log of both sides:

$$\begin{aligned}\ln y &= \ln \frac{\sqrt{x+13}}{(x-4)(\sqrt[3]{2x+1})} \\ &= \frac{1}{2}\ln(x+3) - \ln(x-4) - \frac{1}{3}\ln(2x+1).\end{aligned}$$

Then differentiate both sides,

$$\frac{y'}{y} = \frac{1}{2(x+3)} - \frac{1}{x-4} - \frac{2}{3(2x+1)},$$

and solve for y:

$$y' = \frac{\sqrt{x+13}}{(x-4)(\sqrt[3]{2x+1})}\left(\frac{1}{2(x+3)} - \frac{1}{x-4} - \frac{2}{3(2x+1)}\right).$$

1b. If $y = \tan^{-1} x$, then

$$\tan y = x.$$

Differentiate both sides,

$$(\sec^2 y)y' = 1,$$

and solve for y'

$$y' = \frac{1}{\sec^2 y}.$$

Finally, recal that $\sec^2 y = 1 + \tan^2 y$, so

$$y' = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

2a. Complete the square:

$$\int \frac{1}{x^2+2x+10}dx = \int \frac{1}{x^2+2x+1+9}dx = \int \frac{1}{(x+1)^2+3^2}dx$$

Let $u = x+1$, then the above integral becomes:

$$\int \frac{1}{u^2+3^2}du = \int \frac{1/3^2}{(u/3)^2+1}du.$$

Now let $v = u/3$, then a substitution yields

$$\int \frac{1/3}{v^2+1}dv = \frac{1}{3}\tan^{-1} v + C = \frac{1}{3}\tan^{-1}\left(\frac{x+1}{3}\right) + C.$$

Alternatively, you could just recall that $\int \frac{1}{u^2+a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$.

2b. To integrate $\int \cos^n x\,dx$, where n is an odd integer, first write the inetegrand as $\cos^{n-1} x \cos x$

$$\int \cos^5 x dx = \int \cos^4 x \cos x dx.$$

Then using the formula $\cos^2 x = 1 - \sin^2 x$, the above becomes

$$\int (1-\sin^2 x)^2 \cos x dx = \int (1 - 2\sin x + \sin^2 x)\cos x dx.$$

Now use the substitution $u = \sin x$, to get

$$\begin{aligned}\int (1-2u+u^2)du &= u - u^2 + \frac{1}{3}u^3 + C \\ &= \sin x - \sin^2 x + \frac{1}{3}\sin^3 x + C.\end{aligned}$$

2c. Integrate by parts: let

$$u = \tan^{-1} x \quad \text{and} \quad dv = dx,$$

then

$$du = \frac{1}{1+x^2}dx \quad \text{and} \quad v = x.$$

So
$$\int \tan^{-1} x dx = x \tan^{-1} x + \int \frac{x}{1+x^2} dx + C.$$
The integral on the right is easily computed via the substitution $u = 1+x^2$ to yield the final answer:
$$x \tan^{-1} x + \frac{1}{2} \ln(1+x^2) dx + C.$$

3a. $\lim_{x\to 0}(1+x)^{\frac{1}{x}}$ is indeterminate of the form 1^∞, so begin by setting $y = (1+x)^{\frac{1}{x}}$ and then taking the natural log of both sides:
$$\ln y = \frac{1}{x} \ln(1+x).$$
Now take the limit of both sides as $x \to 0$:
$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln(x+1)}{x}.$$
Since the limit on the right hand side is of the form $0/0$, we may use the L'Hopital's rule:
$$\lim_{x\to 0} \frac{\ln(x+1)}{x} = \lim_{x\to 0} \frac{1/(x+1)}{1} = 1.$$
So $\lim_{x\to 0} \ln y = 1$, but, since ln is continuous at 1, $\lim_{x\to 0} \ln y = \ln(\lim_{x\to 0} y)$. So we have
$$\ln(\lim_{x\to 0} y) = 1,$$
which yields
$$\lim_{x\to 0} y = e^1 = e.$$

3b. $\lim_{x\to 0}(x^2 \ln x)$ is indeterminate of the form $0 \cdot \infty$, so we rewrite is as follows to make it susceptible to L'Hopital's rule:
$$\lim_{x\to 0}(x^2 \ln x) = \lim_{x\to 0} \frac{\ln x}{\frac{1}{x^2}}.$$

Now the limit on the right is of the form ∞/∞, so we may use the L'Hopital's rule:

$$\lim_{x\to 0}\frac{\ln x}{\frac{1}{x^2}}=\lim_{x\to 0}\frac{\frac{1}{x}}{\frac{-2}{x^3}}=\lim_{x\to 0}\frac{x^2}{-2}=0.$$

4.

$$\begin{aligned}4.122222\ldots &= 4.1+0.02+0.002+\cdots\\ &= \frac{41}{10}+\frac{2/100}{1-1/10}\\ &= \frac{41}{10}+\frac{2}{90}\\ &= \frac{281}{90}.\end{aligned}$$

5a.

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\cdots=1+\sum_{n=1}^{\infty}\frac{1}{2n}=1+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$$

But $\sum_{n=1}^{\infty}\frac{1}{n}$ is the harmonic series, which diverges (e.g., by the integral test). So the series on the left converges as diverges as well.

5b. Using the L'Hopital's rule we have

$$\lim_{n\to\infty}\frac{n}{\ln n}=\lim_{n\to\infty}\frac{1}{1/n}=\frac{1}{0}=\infty\neq 0.$$

So by the n^{th}-*term test*, a.k.a. *divergence test*, $\sum_{n=2}^{\infty}\frac{n}{\ln n}$ diverges.

5c. Compare $\sum_{n=1}^{\infty}\frac{n+7}{n^2\sqrt{n}}$ with the series $\sum_{n=1}^{\infty}\frac{n}{n^2\sqrt{n}}$, via the *limit comparison test*:

$$\lim_{n\to\infty}\frac{\frac{n+7}{n^2\sqrt{n}}}{\frac{n}{n^2\sqrt{n}}}=\lim_{n\to\infty}\frac{n+7}{n}=1.$$

So the series either both converge or both diverge. But $\frac{n}{n^2\sqrt{n}}=\frac{n}{n^{5/2}}=\frac{1}{n^{3/2}}$, and $3/2>1$, so by the *p-series test*, the second series converges. Thus the first series converges as well.

5d.

$$\lim_{n\to\infty} \frac{\frac{(n+1)^2}{(n+1)!}}{\frac{n^2}{n!}} = \lim_{n\to\infty} \frac{(n+1)^2 n!}{(n+1)! n^2} = \lim_{n\to\infty} \frac{(n+1)^2}{(n+1)n^2} = \lim_{n\to\infty} \frac{n+1}{n^2} = 0 \leq 1$$

Therefore by the *absolute ratio test* the series convereges.

6. The general term for the series $1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$ is x^n/n (for our purposes here we may safely disregard the first term).

$$\rho(x) := \lim_{n\to\infty} \frac{\frac{x^{n+1}}{(n+1)}}{\frac{x^n}{n}} = \lim_{n\to\infty} \frac{xn}{n+1} = x.$$

For the series to converge we must have $|\rho(x)| < 1$ and solve for x. So in this case we simply get $|x| < 1$ which yields $-1 < x < 1$. So the radius of convergence is 1. Next we check the end points. If $x = 1$, then out series becomes the harmonic series which diverges, but if $x = -1$ then we obtain the alternating harmonic series which converges. Thus the interval convergenc is:

$$[-1, 1).$$

7a. First recall that the summation formula for geometric series yields:

$$1 - x + x^2 - x^3 + \cdots = \frac{1}{1+x}$$

for all $-1 < x < 1$. So, over this interval, we may integate both sides of the above to get:

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \ln(1+x)$$

Since the series on the right converges for $x = 1$ (which is the alternating harmonic series), and $\ln(1 + x)$ is continuous at $x = 1$, then, by Abel's theorem, the above formula holds for $x = 1$. So we get

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln(2).$$

7b. The summation formula for geometric series yields:

a) Find an infinite series which converges to π (*Hint*: Find a power series for $\tan^{-1} x$).

c) Find $\lim_{n\to\infty} \frac{x^n}{n!}$ (*Hint*: consider the series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$).

Problems 2 and 5 are worth 20 points and 40 points respectively; the rest are worth 10 points each