Solutions To Midterm 3

1a. This is best done using *logarithmic differentiation*. First take natural log of both sides:

\[
\ln y = \ln \frac{\sqrt{x + 13}}{(x - 4)(\sqrt{2x + 1})} \\
= \frac{1}{2} \ln(x + 3) - \ln(x - 4) - \frac{1}{3} \ln(2x + 1).
\]

Then differentiate both sides,

\[
\frac{y'}{y} = \frac{1}{2(x + 3)} - \frac{1}{x - 4} - \frac{2}{3(2x + 1)},
\]

and solve for \( y' \):

\[
y' = \frac{\sqrt{x + 13}}{(x - 4)(\sqrt{2x + 1})} \left( \frac{1}{2(x + 3)} - \frac{1}{x - 4} - \frac{2}{3(2x + 1)} \right).
\]

1b. If \( y = \tan^{-1} x \), then

\[\tan y = x.\]

Differentiate both sides,

\[(\sec^2 y)y' = 1,\]

and solve for \( y' \)

\[y' = \frac{1}{\sec^2 y}.\]

Finally, recall that \( \sec^2 y = 1 + \tan^2 y \), so

\[y' = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}.\]
2a. Complete the square:

\[ \int \frac{1}{x^2 + 2x + 10} \, dx = \int \frac{1}{x^2 + 2x + 1 + 9} \, dx = \int \frac{1}{(x + 1)^2 + 3^2} \, dx \]

Let \( u = x + 1 \), then the above integral becomes:

\[ \int \frac{1}{u^2 + 3^2} \, du = \int \frac{1/3^2}{(u/3)^2 + 1} \, du. \]

Now let \( v = u/3 \), then a substitution yields

\[ \int \frac{1/3}{v^2 + 1} \, dv = \frac{1}{3} \tan^{-1} v + C = \frac{1}{3} \tan^{-1} \left( \frac{x + 1}{3} \right) + C. \]

Alternatively, you could just recall that \( \int \frac{1}{u^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C. \)

2b. To integrate \( \int \cos^n x \, dx \), where \( n \) is an odd integer, first write the integrand as \( \cos^{n-1} x \cos x \)

\[ \int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx. \]

Then using the formula \( \cos^2 x = 1 - \sin^2 x \), the above becomes

\[ \int (1 - \sin^2 x)^2 \cos x \, dx = \int (1 - 2\sin x + \sin^2 x) \cos x \, dx. \]

Now use the substitution \( u = \sin x \), to get

\[ \int (1 - 2u + u^2) \, du = u - u^2 + \frac{1}{3} u^3 + C \]
\[ = \sin x - \sin^2 x + \frac{1}{3} \sin^3 x + C. \]

2c. Integrate by parts: let

\[ u = \tan^{-1} x \quad \text{and} \quad dv = dx, \]

then

\[ du = \frac{1}{1 + x^2} \, dx \quad \text{and} \quad v = x. \]
So
\[ \int \tan^{-1} x \, dx = x \tan^{-1} x + \int \frac{x}{1 + x^2} \, dx + C. \]

The integral on the right is easily computed via the substitution \( u = 1 + x^2 \)
to yield the final answer:
\[ x \tan^{-1} x + \frac{1}{2} \ln(1 + x^2) \, dx + C. \]

3a. \( \lim_{x \to 0} (1 + x)^{\frac{1}{2}} \) is indeterminate of the form \( 1^\infty \), so begin by setting \( y = (1 + x)^{\frac{1}{2}} \) and then taking the natural log of both sides:
\[ \ln y = \frac{1}{x} \ln(1 + x). \]

Now take the limit of both sides as \( x \to 0 \):
\[ \lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln(x + 1)}{x}. \]

Since the limit on the right hand side is of the form \( 0/0 \), we may use the L'Hôpital’s rule:
\[ \lim_{x \to 0} \frac{\ln(x + 1)}{x} = \lim_{x \to 0} \frac{1/(x + 1)}{1} = 1. \]

So \( \lim_{x \to 0} \ln y = 1 \), but, since \( \ln \) is continuous at 1, \( \lim_{x \to 0} \ln y = \ln(\lim_{x \to 0} y) \).

So we have
\[ \ln(\lim_{x \to 0} y) = 1, \]
which yields
\[ \lim_{x \to 0} y = e^1 = e. \]

3b. \( \lim_{x \to 0} (x^2 \ln x) \) is indeterminate of the form \( 0 \cdot \infty \), so we rewrite is as follows to make it susceptible to L'Hôpital’s rule:
\[ \lim_{x \to 0} (x^2 \ln x) = \lim_{x \to 0} \frac{\ln x}{x^2}. \]
Now the limit on the right is of the form \( \infty / \infty \), so we may use the L'Hopital's rule:

\[
\lim_{x \to 0} \frac{\ln x}{x^2} = \lim_{x \to 0} \frac{\frac{1}{x}}{2x} = \lim_{x \to 0} \frac{x^2}{2} = 0.
\]

4.

\[
4.12222\ldots = 4.1 + 0.02 + 0.002 + \cdots = \frac{41}{10} + \frac{2}{100} = \frac{41}{10} + \frac{2}{1 - 1/10} = \frac{41}{10} + \frac{2}{90} = \frac{281}{90}.
\]

5a.

\[
1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots = 1 + \sum_{n=1}^{\infty} \frac{1}{2n} = 1 + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}.
\]

But \( \sum_{n=1}^{\infty} \frac{1}{n} \) is the harmonic series, which diverges (e.g., by the integral test). So the series on the left converges as diverges as well.

5b. Using the L'Hopital's rule we have

\[
\lim_{n \to \infty} \frac{n}{\ln n} = \lim_{n \to \infty} \frac{1}{1/n} = \infty \neq 0.
\]

So by the \( n^{th} \)-term test, a.k.a. divergence test, \( \sum_{n=2}^{\infty} \frac{n}{\ln n} \) diverges.

5c. Compare \( \sum_{n=1}^{\infty} \frac{n+7}{n^{3/7}} \) with the series \( \sum_{n=1}^{\infty} \frac{n}{n^{3/7}} \), via the limit comparison test:

\[
\lim_{n \to \infty} \frac{\frac{n+7}{n^{3/7}}}{\frac{n}{n^{3/7}}} = \lim_{n \to \infty} \frac{n+7}{n} = 1.
\]

So the series either both converge or both diverge. But \( \frac{n}{n^{3/7}} = \frac{n}{n^{3/7}} = \frac{1}{n^{3/7}} \), and \( 3/2 > 1 \), so by the \( p \)-series test, the second series converges. Thus the first series converges as well.
5d. 
\[
\lim_{n \to \infty} \frac{\frac{(n+1)^2}{n!}}{\frac{n^2}{n!}} = \lim_{n \to \infty} \frac{(n+1)^2}{(n+1)!n^2} = \lim_{n \to \infty} \frac{(n+1)^2}{n+1} = \lim_{n \to \infty} \frac{n+1}{n^2} = 0 \leq 1
\]

Therefore by the absolute ratio test the series converges.

6. The general term for the series \(1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\) is \(x^n/n\) (for our purposes here we may safely disregard the first term).

\[
\rho(x) := \lim_{n \to \infty} \frac{x^{n+1}}{(n+1) x^n} = \lim_{n \to \infty} \frac{x n}{n+1} = x.
\]

For the series to converge we must have \(|\rho(x)| < 1\) and solve for \(x\). So in this case we simply get \(|x| < 1\) which yields \(-1 < x < 1\). So the radius of convergence is 1. Next we check the end points. If \(x = 1\), then out series becomes the harmonic series which diverges, but if \(x = -1\) then we obtain the alternating harmonic series which converges. Thus the interval convergence is:

\([-1, 1)\).

7a. First recall that the summation formula for geometric series yields:

\[
1 - x + x^2 - x^3 + \cdots = \frac{1}{1 + x}
\]

for all \(-1 < x < 1\). So, over this interval, we may integrate both sides of the above to get:

\[
x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \ln(1 + x)
\]

Since the series on the right converges for \(x = 1\) (which is the alternating harmonic series), and \(\ln(1 + x)\) is continuous at \(x = 1\), then, by Abel’s theorem, the above formula holds for \(x = 1\). So we get

\[
1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln(2).
\]

7b. The summation formula for geometric series yields:
a) Find an infinite series which converges to $\pi$ (Hint: Find a power series for $\tan^{-1} x$).

c) Find $\lim_{n \to \infty} \frac{x^n}{n!}$ (Hint: consider the series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$).

Problems 2 and 5 are worth 20 points and 40 points respectively; the rest are worth 10 points each.