Math 23B
Multivariable Calculus
Fall 1999, UCSC

**FINAL EXAM**

Time: 180 min

1. Evaluate \( \int \int_D \sin(x^2 + y^2) \, dx \, dy \) where \( D \) is the disk \( x^2 + y^2 \leq \pi \).

2. Compute the volume of an ellipsoid with semiaxes \( a, b, \) and \( c \).

3. Find the center of mass of the ice cream cone given by \( x^2 + y^2 + z^2 \leq 1 \) and \( z \geq \sqrt{x^2 + y^2} \), if the density is \( \delta(x, y, z) := \sqrt{x^2 + y^2 + z^2} \).

4. Find the average value of the distance of the helix \( (\cos t, \sin t, t) \), \( 0 \leq t \leq \pi \), from the \( xz \)-plane.

5. Find the surface area of the portion of the sphere \( x^2 + y^2 + z^2 = 1 \) which lies above the plane \( z = \frac{\sqrt{2}}{2} \).

6. Show that the gravitational vectorfield \( \mathbf{F} := -\frac{\mathbf{r}}{\|\mathbf{r}\|^3} \) is conservative. What is the total work done by this force in moving a particle from a point \( \mathbf{r}_0 \) to a point \( \mathbf{r}_1 \)?

7. Compute the area of the region enclosed by the curve \( (\cos^3 t, \sin^3 t), \) \( 0 \leq t \leq 2\pi \). (Hint: use Green's theorem).

8. Show that the volume of any cone with base \( D \) and height \( h \) is given by \( 1/3 \text{Area}(D)h \) (Hint: suppose that the vertex of the cone is at the origin and apply Gauss's theorem to the vectorfield \( \mathbf{r} := (x, y, z) \)).

9. Find \( \int \int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} \) where \( S \) is given by \( x^2 + y^2 + (z - \frac{1}{2})^2 = 1, \) and \( z \geq 0, \) and \( \mathbf{F}(x, y, z) := (x + z, y + z, z^2) \) (Hint: use Stokes's theorem).

10. Suppose that rain is described by the vectorfield \( \mathbf{F}(x, y, z) = -(1, 0, 1). \) What is the flux through the hemispherical cup \( z = -\sqrt{1 - x^2 - y^2} \)? How long does it take before the cup is filled with water?

11 (Bonus). Prove that the there exists a surface which has infinite area but bounds a finite volume.

*Each problem is worth 10 points.*