Math 426 Jan 27, 2004[1]
Introduction to Modern Geometry
Spring 2004, PSU

Lecture Notes 1

1 Curves

1.1 Definition, Examples, and Reparametrization

A (parametrized) *curve* (in Euclidean space) is a mapping $\alpha\colon I \to \mathbf{R}^n$, where I is an interval in the real line, and $\mathbf{R}^n$ is the set of all n-tuples of real numbers. We also use the notation

$$I \ni t \xmapsto{\alpha} \alpha(t) \in \mathbf{R}^n,$$

which emphasizes that α sends each element of the interval I to a certain point in $\mathbf{R}^n$. We shall always assume that α is continuous, and whenever we need to differentiate it we will assume that α is differentiable up to however many orders that we may need.

Some standard examples of curves are a *line* which passes through a point $p \in \mathbf{R}^n$, is parallel to the vector $v \in \mathbf{R}^n$, and has constant speed $\|v\|$

$$[0, 2\pi] \ni t \xmapsto{\alpha} p + tv \in \mathbf{R}^n;$$

a *cricle* of radius $\mathbf{R}$ in the plane, which is oriented counterclockwise,

$$[0, 2\pi] \ni t \xmapsto{\alpha} \big(r\cos(t), r\sin(t)\big) \in \mathbf{R}^2;$$

and the right handed *helix* (or corckscrew) given by

$$\mathbf{R} \ni t \xmapsto{\alpha} \big(r\cos(t), r\sin(t), t\big) \in \mathbf{R}^3.$$

Other famous examples include the *parabola*

$$\mathbf{R} \ni t \xmapsto{\alpha} \big(t, t^2\big) \in \mathbf{R}^2,$$

and the *cubic curve*

$$\mathbf{R} \ni t \xmapsto{\alpha} \big(t, t^2, t^3\big) \in \mathbf{R}^3.$$

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Exercise 1. Sketch the cubic curve (*Hint:* First draw each of the projections into the xy, yz, and zx planes).

Exercise 2. Find a formula for the curve which is traced by the motion of a fixed point on a wheel of radius r rolling with constant speed on a flat surface (*Hint:* Add the formula for a circle to the formula for a line generated by the motion of the center of the wheel. You only need to make sure that the speed of the line correctly matches the speed of the circle).

Exercise 3. Let $\alpha\colon I \to \mathbf{R}^n$, and $\beta\colon J \to \mathbf{R}^n$ be a pair of differentiable *curves*. Show that

$$\Big(\langle\alpha(t),\beta(t)\rangle\Big)' = \langle\alpha'(t),\beta(t)\rangle + \langle\alpha(t),\beta'(t)\rangle$$

and

$$\Big(\|\alpha(t)\|\Big)' = \frac{\langle\alpha(t),\alpha'(t)\rangle}{\|\alpha(t)\|}.$$

(*Hint:* The first identity follows immediately from the definition of the inner-product, together with the ordinary product rule for derivatives. The second identity follows from the first once we recall that $\|\cdot\| := \langle\cdot,\cdot\rangle^{1/2}$).

Exercise 4. Show that if α has unit speed, i.e., $\|\alpha'(t)\| = 1$, then its velocity and acceleration are orthogonal, i.e., $\langle\alpha(t),\alpha''(t)\rangle = 0$.

Exercise 5. Show that if the position vector and velocity of a planar curve $\alpha\colon I \to \mathbf{R}^2$ are always perpendicular, i.e., $\langle\alpha(t),\alpha'(t)\rangle = 0$, for all $t \in I$, then $\alpha(I)$ lies on a circle centered at the origin of $\mathbf{R}^2$.

We say that β is a *reparametrization* of α provided that there exists a smooth bijection $\theta\colon I \to J$ such that

$$\alpha(t) = \beta(\theta(t)).$$

For instance $\beta(t) = (\cot(2t), \sin(2t))$, $0 \le t \le \pi$, is a reparametrization $\alpha(t) = (\sin(t), \cos(t))$, $0 \le t \le 2\pi$, with $\theta\colon [0, 2\pi] \to [0, \pi]$ given by $\theta(t) = t/2$.

The *geometric quantities* associated to a curve, are those quantities that do not change under reparametrization. These include length and curvature as we define below.

1.2 Length

The *length* of α is defined as

$$\text{length}[\alpha] := \int_I \|\alpha'(t)\|\, dt.$$

One can show, using the definition of integral and the mean value theorem, that the above is the limit of the length polygonal curves which converge to α.

Exercise 6. Compute the length of a circle of radius r.

Exercise 7. Show that if β is a reparametrization of α, then $\text{length}[\beta] = \text{length}[\alpha]$, i.e., length is invariant under reparametrization (*Hint*: you only need to recall the chain rule together with the integration by substitution)

Let $L := \text{length}[\alpha]$. The *arclength* function of α is a mapping $s\colon [a, b] \to [0, L]$ given by

$$s(t) := \int_a^t \|\alpha'(u)\|\, du.$$

Thus $s(t)$ is the length of the subsegment of α which stretches from the inital time a to time t.

Exercise 8. Show that if α is a *regular* curve, i.e., $\|\alpha'(t)\| \neq 0$ for all $t \in I$, then $s(t)$ is an invertible function, i.e., it is one-to-one (*Hint* compute $s'(t)$).

Exercise 9. Show that every regular curve $\alpha\colon [a, b] \to \mathbf{R}^n$, may be reparametrized by arclength (**Hint:** Define $\beta\colon [0, L] \to \mathbf{R}^n$ by $\beta(t) := \alpha(s^{-1}(t))$, and use the chain rule to show that $\|\beta'\| = 1$; you also need to recall that since $f(f^{-1}(t)) = t$, then, again by chain rule, we have $(f^{-1})'(t) = 1/f'(f^{-1}(t))$ for any smooth function f with nonvanishing derivative.)