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Introduction to Modern Geometry
Spring 2004, PSU

Lecture Notes 10

2.3 Meaning of Gaussian Curvature

In the previous lecture we gave a formal definition for Gaussian curvature K in terms of the differential of the gauss map, and also derived explicit formulas for K in local coordinates. In this lecture we explore the geometric meaning of K.

2.3.1 A measure for local convexity

Let $M \subset \mathbf{R}^3$ be a regular embedded surface, $p \in M$, and H_p be hyperplane passing through p which is parallel to T_pM. We say that M is *locally convex* at p if there exists an open neighborhood V of p in M such that V lies on one side of H_p. In this section we prove:

Theorem 1. *If $K(p) > 0$ then M is locally convex at p, and if $k(p) < 0$ then M is not locally convex at p.*

When $K(p) =$, we cannot in general draw an conclusion with regard to the local convexity of M at p as the following two exercises demonstrate:

Exercise 2. Show that there exists a surface M and a point $p \in M$ such that M is strictly locally convex at p; however, $K(p) = 0$ (*Hint:* Let M be the graph of the equation $z = (x^2+y^2)^2$. Then may be covered by the Monge patch $X(u^1, u^2) := (u^1, u^2, ((u^1)^2 + (u^2))^2)$. Use the Monge Ampere equation derived in the previous lecture to compute the curvature at $X(0, 0)$.).

Exercise 3. Let M be the *Monkey saddle*, i.e., the graph of the equation $z = y^3 - 3yx^2$, and $p := (0, 0, 0)$. Show that $K(p) = 0$, but M is not locally convex at p.

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After a rigid motion we may assume that $p = (0,0,0)$ and T_pM is the xy-plane. Then, using the inverse function theorem , it is easy to show that there exists a Monge Patch (U, X) centered at p, as the follwing exercise demonstrates:

Exercise 4. Define $\pi \colon M \to \mathbf{R}^2$ by $\pi(q) := (q^1, q^2, 0)$. Show that $d\pi_p$ is locally one-to-one. Then, by the inverse function theorem, it follows that π is a local diffeomorphism. So there exists a neighborhood U of $(0,0)$ such that $\pi^{-1} \colon U \to M$ is one-to-one and smooth. Let $f(u^1, y^2)$ denote the z-coordinate of $\pi^{-1}(u^1, u^2)$, and set $X(u^1, u^2) := (u^1, u^2, f(u^1, u^2))$. Show that (U, X) is a proper regular patch.

The previous exercisle shows that local convexity of M at p depends on whether or not f changes sign in a neighborhood of the origin. To examine this we need to recall the Taylor's formula for functions of two variables:

$$f(u_1, u_2) = f(0,0) + \sum_{i=1}^{2} D_i f(0,0) + \frac{1}{2}\sum_{i,j=1}^{2} D_{ij}(\xi^1, \xi^2) u^i u^j,$$

where (ξ^1, ξ^2) is a point on the line connecting (u^1, u^2) to $(0,0)$.

Exercise 5. Prove the Taylor's formula given above. (*Hints*: First recall Taylor's formula for functions of one variable: $g(t) = g(0) + g'(0)t + (1/2)g''(s)t^2$, where $s \in [0,t]$. Then define $\gamma(t) := (tu^1, tu^2)$, set $g(t) := f(\gamma(t))$, and apply Taylor's formula to g. Then chain rule will yield the desired result.)

Next note that, by construction, $f(0,0) = 0$. Further $D_1 f(0,0) = 0 = D_2 f(0,0)$ as well. Thus

$$f(u_1, u_2) = \frac{1}{2}\sum_{i,j=1}^{2} D_{ij}(\xi^1, \xi^2) u^i u^j.$$

Hence to complete the proof of Theorem 1, it remains to show how the quanitity on the right hand side of the above euation is influence by $K(p)$. To this end, recall the Monge-Ampere equation for curvature:

$$\det(\operatorname{Hess} f\big(\xi^1, \xi^2)\big) = K(f(\xi^1, \xi^2))\big(1 + \| \operatorname{grad} f(\xi^1, \xi^2)\|^2\big)^2.$$

Now note that $K(f(0,0)) = K(p)$. Thus, by continuity, if U is a sufficiently small neighborhood of $(0,0)$, the sign of $\det(\operatorname{Hess} f)$ agrees with the sign of $K(p)$ throughout U.

Finally, we need some basic facts about quadratic forms. A *quadratic form* is a function of two variables $Q\colon \mathbf{R}^2 \to \mathbf{R}$ given by

$$Q(x,y) = ax^2 + 2bxy + cy^2,$$

where a, b, and c are constants. Q is said to be definite if $Q(x,x) \neq 0$ whenver $x \neq 0$.

Exercise 6. Show that if $ac - b^2 > 0$, then Q is definite, and if $ac - b^2 < 0$, then Q is not definite. (*Hints*: For the first part, suppose that $x \neq 0$, but $Q(x,y) = 0$. Then $ax^2+2bxy+cy^2 = 0$, which yields $a+2b(x/y)+c(x/y)^2 = 0$. Thus the discriminant of this equation must be positive, which will yield a contradiction. The proof of the second part is similar).

Theorem 1 follows from the above exercise.

2.3.2 Ratio of areas

In the previous subsection we gave a geometric interpretation for the sign of Gaussian curvature. Here we describe the geometric significance of the magnitude of K.

If V is a sufficiently small neighborhood of p in M (where M, as always, denotes a regular embedded surface in $\mathbf{R}^3$), then it is easy to show that there exist a patch (U, X) centered at p such that $X(U) = V$. Area of V is then defined as follows:

$$\operatorname{Area}(V) := \int\int_U \|D_1X \times D_2X\|\, du^1 du^2.$$

Using the chain rule, one can show that the above definition is independent of the the patch.

Exercise 7. Let $V \subset \mathbf{S}^2$ be a region bounded in between a pair of great circles meeting each other at an angle of α. Show that $\operatorname{Area}(V) = 2\alpha$(*Hints*: Let $U := [0,\alpha] \times [0,\pi]$ and $X(\theta,\phi) := (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$. Show that $\|D_1X \times D_2X\| = |\sin\phi|$. Further, note that, after a rotation we may assume that $X(U) = V$. Then an integration will yield the desired result).

Exercise 8. Use the previous exercise to show that the area of a geodesic triangle $T \subset \mathbf{S}^2$ (a region bounded by three great circles) is equal to sum of its angles minus π (*Hints:* Use the picture below: $A+B+C+T=2\pi$, and $A=2\alpha-T$, $B=2\beta-T$, and $C=2\gamma-T$).

Let $V_r := B_r(p)\cap M$. Then, if r is sufficiently small, $V(r)\subset X(U)$, and, consequently, $U_r := X^{-1}(V_r)$ is well defined. In particular, we may compute the area of V_r using the patch (U_r, X). In this section we show that

$$|K(p)| = \lim_{r\to 0}\frac{\text{Area}(n(V_r))}{\text{Area}(V_r)}.$$

Exercise 9. Recall that the mean value theorem states that $\int\int_U f du^1 du^2 = f(\bar{u}^1,\bar{u}^2)\,\text{Area}(U)$, for some $(\bar{u}^1,\bar{u}^2)\in U$. Use this theorem to show that

$$\lim_{r\to 0}\frac{\text{Area}(n(V_r))}{\text{Area}(V_r)} = \frac{\|D_1N(0,0)\times D_2N(0,0)\|}{\|D_1X(0,0)\times D_2X(0,0)\|}$$

(Recall that $N := n\circ X$.)

Exercise 10. Prove Lagrange's identity: for every pair of vectors $v, w\in\mathbf{R}^3$,

$$\|v\times w\|^2 = \det\begin{vmatrix} \langle v,v\rangle & \langle v,w\rangle \\ \langle w,v\rangle & \langle w,w\rangle \end{vmatrix}.$$

Now set $g(u^1,u^2) := \det[g_{ij}(u^1,u^2)]$. Then, by the previous exercise it follows that $\|D_1X(0,0)\times D_2X(0,0)\| = \sqrt{g(0,0)}$. Hence, to complete the proof of the main result of this section it remains to show that

$$\|D_1N(0,0)\times D_2N(0,0)\| = K(p)\sqrt{g(0,0)}.$$

We prove the above formula using two different methods:

METHOD 1. Recall that $K(p) := \det(S_p)$, where $S_p := -dn_p \colon T_pM \to T_pM$ is the shape operator of M at p. Also recall that $D_iX(0,0)$, $i = 1$, 2, form a basis for T_pM. Let S_{ij} be the coefficients of the matrix representation of S_p with respect to this basis, then

$$S_p(D_iX) = \sum_{j=1}^{2} S_{ij}\, D_jX.$$

Further, recall that $N := n \circ X$. Thus the chain rule yields:

$$S_p(D_iX) = -dn(D_iX) = -D_i(n \circ X) = -D_iN.$$

Exercise 11. Verify the middle step in the above formula, i.e., show that $dn(D_iX) = D_i(n \circ X)$.

From the previous two lines of formulas, it now follows that

$$-D_iN = \sum_{j=1}^{2} S_{ij}\, D_jX.$$

Taking the inner product of both sides with D_kN, $k = 1$, 2, we get

$$\langle -D_iN, D_kN\rangle = \sum_{j=1}^{2} S_{ij}\langle D_jX, D_kN\rangle.$$

Exercise 12. Let F, $G \colon U \subset \mathbf{R}^2 \to \mathbf{R}^3$ be a pair of mappings such that $\langle F, G\rangle = 0$. Prove that $\langle D_iF, G\rangle = -\langle F, D_iG\rangle$.

Now recall that $\langle D_jX, N\rangle = 0$. Hence the previous exercise yields:

$$\langle D_jX, D_kN\rangle = -\langle D_{kj}X, N\rangle = -l_{ij}.$$

Combining the previous two lines of formulas, we get: $\langle D_iN, D_kN\rangle = \sum_{k=1}^{2} S_{ij}l_{jk}$; which in matrix notation is equivalent to

$$[\langle D_iN, D_jN\rangle] = [S_{ij}][l_{ij}].$$

Finally, recall that $\det[\langle D_iN, D_kN\rangle] = \|D_1N \times D_2N\|^2$, $\det[S_{ij}] = K$, and $\det[l_{ij}] = Kg$. Hence taking the determinant of both sides in the above equation, and then taking the square root yields the desired result.

Next, we discuss the second method for proving that $\|D_1 N \times D_2 N\| = K\sqrt{g}$.

METHOD 2. Here we work with a special patch which makes the computations easier:

Exercise 13. Show that there exist a patch (U, X) centered at p such that $[g_{ij}(0,0)]$ is the identity matrix. (*Hint:* Start with a Monge patch with respect to T_pM)

Thus, if we are working with the coordinate patch referred to in the above exercise, $g(0,0) = 1$, and, consequently, all we need is to prove that $\|D_1 N(0,0) \times D_2 N(0,0)\| = K(p)$.

Exercise 14. Let $f\colon U \subset \mathbf{R}^2 \to \mathbf{S}^2$ be a differentiable mapping. Show that $\langle D_i f(u^1, u^2), f(u^1, u^2)\rangle = 0$ (*Hints:* note that $\langle f, f\rangle = 1$ and differentiate).

It follows from the previous exercise that $\langle D_i N, N\rangle = 0$. Now recall that $N(0,0) = n \circ X(0,0) = n(p)$. Hence, we may conclude that $N(0,0) \in T_pM$. Further recall that $\{D_1 X(0,0), D_2 X(0,0)\}$ is now an orthonormal basis for T_pM (because we have chosen (U, X) so that $[g_{ij}(0,0)]$ is the identity matrix). Consequently,

$$D_i N = \sum_{k=1}^{2} \langle D_i N, D_k X\rangle D_k X,$$

where we have omitted the explicit reference to the point $(0,0)$ in the above formula in order to make the notation less cumbersome (it is important to keep in mind, however, that the above is valid only at $(0,0)$). Taking the inner product of both sides of this equation with $D_j N(0,0)$ yields:

$$\langle D_i N, D_j N\rangle = \sum_{k=1}^{2} \langle D_i N, D_k X\rangle \langle D_k X, D_j N\rangle.$$

Now recall that $\langle D_i N, D_k X\rangle = -\langle N, D_{ij} X\rangle = -l_{ij}$. Similarly, $\langle D_k X, D_j N\rangle = -l_{kj}$. Thus, in matrix notation, the above formula is equivalent to the following:

$$[\langle D_i N, D_j N\rangle] = [l_{ij}]^2$$

Finally, recall that $K(p) = \det[l_{ij}(0,0)]/\det[g_{ij}(0,0)] = \det[l_{ij}(0,0)]$. Hence, taking the determinant of both sides of the above equation yields the desired result.

2.3.3 Product of principal curvatures

For every $v \in T_pM$ with $\|v\| = 1$ we define the *normal curvature* of M at p in the direction of v by

$$k_v(p) := \langle \gamma''(0), n(p) \rangle,$$

where $\gamma\colon (-\epsilon, \epsilon) \to M$ is a curve with $\gamma(0) = p$ and $\gamma'(0) = v$.

Exercise 15. Show that $k_v(p)$ does not depend on γ.

In particular, by the above exercise, we may take γ to be a curve which lies in the intersection of M with a plane which passes through p and is normal to $n(p) \times v$. So, intuitively, $k_v(p)$ is a measure of the curvature of an orthogonal cross section of M at p.

Let $UT_pM := \{v \in T_pM \mid \|v\| = 1\}$ denote the *unit tangent space* of M at p. The *principal curvatures* of M at p are defined as

$$k_1(p) := \min_v k_v(p), \quad \text{and} \quad k_2(p) := \max_v k_v(p),$$

where v ranges over UT_pM. Our main aim in this subsection is to show that

$$K(p) = k_1(p)k_2(p).$$

Since $K(p)$ is the determinant of the shape operator S_p, to prove the above it suffices to show that $k_1(p)$ and $k_2(p)$ are the eigenvalues of S_p.

First, we need to define the *second fundamental form* of M at p. This is a bilinear map $\mathrm{II}_p\colon T_pM \times T_pM \to \mathbf{R}$ defined by

$$\mathrm{II}_p(v, w) := \langle S_p(v), w \rangle.$$

We claim that, for all $v \in UT_pM$,

$$k_v(p) = \mathrm{II}_p(v, v).$$

The above follows from the following computation

$$\begin{aligned}
\langle S_p(v), v \rangle &= -\langle dn_p(v), v \rangle \\
&= -\langle (n \circ \gamma)'(0), \gamma'(0) \rangle \\
&= \langle (n \circ \gamma)(0), \gamma''(0) \rangle \\
&= \langle n(p), \gamma''(0) \rangle
\end{aligned}$$

Exercise 16. Verify the passage from the second to the third line in the above computation, i.e., show that $-\langle (n\circ\gamma)'(0), \gamma'(0)\rangle = \langle (n\circ\gamma)(0), \gamma''(0)\rangle$ (*Hint:* Set $f(t) := \langle n(\gamma(t)), \gamma'(t)\rangle$, note that $f(t) = 0$, and differentiate.)

So we conclude that $k_i(p)$ are the minimum and maximum of $\mathrm{II}_p(v)$ over UT_pM. Hence, all we need is to show that the extrema of II_p over UT_pM coincide with the eigenvalues of S_p.

Exercise 17. Show that II_p is symmetric, i.e., $\mathrm{II}_p(v,w) = \mathrm{II}_p(w,v)$ for all v, $w \in T_pM$.

By the above exercise, S_p is a self-adjoint operator, i.e, $\langle S_p(v), w\rangle = \langle v, S_p(w)\rangle$. Hence S_p is orthogonally diagonalizable, i.e., there exist orthonormal vectors $e_i \in T_pM$, $i = 1, 2$, such that

$$S_p(e_i) = \lambda_i e_i.$$

By convention, we suppose that $\lambda_1 \leq \lambda_2$. Now note that each $v \in UT_pM$ may be represented uniquely as $v = v^1 e_1 + v^2 e_2$ where $(v^1)^2 + (v^2)^2 = 1$. So for each $v \in UT_pM$ there exists a unique angle $\theta \in [0, 2\pi)$ such that

$$v(\theta) := \cos\theta e_1 + \sin\theta e_2;$$

Consequently, bilinearity of II_p yields

$$\mathrm{II}_p(v(\theta), v(\theta)) = \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta.$$

Exercise 18. Verify the above claim, and show that minimum and maximum values of II_p are λ_1 and λ_2 respectively. Thus $k_1(p) = \lambda_1$, and $k_2(p) = \lambda_2$.

The previous exercise completes the proof that $K(p) = k_1(p)k_2(p)$, and also yields the following formula which was discovered by Euler:

$$k_v(p) = k_1(p)\cos^2\theta + k_2(p)\sin^2\theta.$$

In particular, note that by the above formula there exists always a pair of *orthogonal* directions where $k_v(p)$ achieves its maximum and minimum values. These are known as the *principal directions* of M at p.