1.13 The Frenet-Serret Frame and Torsion

Recall that if $\alpha : I \to \mathbb{R}^n$ is a unit speed curve, then the unit tangent vector is defined as

$$T(t) := \alpha'(t).$$

Further, if $\kappa(t) = \|T'(t)\| \neq 0$, we may define the principal normal as

$$N(t) := \frac{T'(t)}{\kappa(t)}.$$

As we saw earlier, in $\mathbb{R}^2$, $\{T, N\}$ form a moving frame whose derivatives may be expressed in terms of $\{T, N\}$ itself. In $\mathbb{R}^3$, however, we need a third vector to form a frame. This is achieved by defining the binormal as

$$B(t) := T(t) \times N(t).$$

Then similar to the computations we did in finding the derivatives of $\{T, N\}$, it is easily shown that

$$\begin{pmatrix} T(t) \\ N(t) \\ B(t) \end{pmatrix}' = \begin{pmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{pmatrix} = \begin{pmatrix} T(t) \\ N(t) \\ B(t) \end{pmatrix},$$

where $\tau$ is the torsion which is defined as

$$\tau(t) := -\langle B', N \rangle.$$

Note that torsion is well defined only when $\kappa \neq 0$, so that $N$ is defined. Torsion is a measure of how much a space curve deviates from lying in a plane:
**Exercise 1.** Show that if the torsion of a curve $\alpha: I \to \mathbb{R}^3$ is zero everywhere then it lies in a plane. (*Hint:* We need to check that there exist a point $p$ and a (fixed) vector $v$ in $\mathbb{R}^3$ such that $\langle \alpha(t) - p, v \rangle = 0$. Let $v = B$, and $p$ be any point of the curve.)

**Exercise 2.** Compute the curvature and torsion of the circular helix

$$(r \cos t, r \sin t, ht)$$

where $r$ and $h$ are constants. How does changing the values of $r$ and $h$ effect the curvature and torsion.

### 1.14 Curves of Constant Curvature and Torsion

The above exercise shows that the curvature and torsion of a circular helix are constant. The converse is also true

**Theorem 3.** *The only curve $\alpha: I \to \mathbb{R}^3$ whose curvature and torsion are nonzero constants is the circular helix.*

The rest of this section develops a number of exercises which leads to the proof of the above theorem

**Exercise 4.** Show that $\alpha: I \to \mathbb{R}^3$ is a circular helix (up to rigid motion) provided that there exists a vector $v$ in $\mathbb{R}^3$ such that

$$\langle T, v \rangle = \text{const},$$

and the projection of $\alpha$ into a plane orthogonal to $v$ is a circle.

So first we need to show that when $\kappa$ and $\tau$ are constants, $v$ of the above exercise can be found. We do this with the aid of the Frenet-Serret frame. Since $\{T, N, B\}$ is an orthonormal frame, we may write

$$v = a(t)T(t) + b(t)N(t) + c(t)B(t).$$

Next we need to find $a$, $b$ and $c$ subject to the conditions that $v$ is a constant vector, i.e., $v' = 0$, and that $\langle T, v \rangle = \text{const}$. The latter implies that

$$a = \text{const}$$

because $\langle T, v \rangle = a$. In particular, we may set $a = 1$. 

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Exercise 5. By setting \( v' = 0 \) show that
\[
v = T + \frac{\kappa}{\tau} B,
\]
and check that \( v \) is the desired vector, i.e. \( \langle T, v \rangle = \text{const} \) and \( v' = 0 \).

So to complete the proof of the theorem, only the following remains:

Exercise 6. Show that the projection of \( \alpha \) into a plane orthogonal to \( v \), i.e.,
\[
\overline{\alpha}(t) := \alpha(t) - \langle \alpha(t), v \rangle \frac{v}{\|v\|^2}
\]
is a circle. (Hint: Compute the curvature of \( \overline{\alpha} \).)

1.15 Intrinsic Characterization of Spherical Curves

In this section we derive a characterization in terms of curvature and torsion for unit speed curves which lie on a sphere. Suppose \( \alpha: I \to \mathbb{R}^3 \) lies on a sphere of radius \( r \). Then there exists a point \( p \) in \( \mathbb{R}^3 \) (the center of the sphere) such that
\[
\|\alpha(t) - p\| = r.
\]
Thus differentiation yields
\[
\langle T(t), \alpha(t) - p \rangle = 0.
\]
Differentiating again we obtain:
\[
\langle T'(t), \alpha(t) - p \rangle + 1 = 0.
\]
The above expression shows that \( \kappa(t) \neq 0 \). Consequently \( N \) is well defined, and we may rewrite the above expression as
\[
\kappa(t) \langle N(t), \alpha(t) - p \rangle + 1 = 0.
\]
Differentiating for the third time yields
\[
k'(t) \langle N(t), \alpha(t) - p \rangle + k(t) \langle -k(t)T(t) + \tau(t)B(t), \alpha(t) - p \rangle = 0,
\]
which using the previous expressions above we may rewrite as
\[
-k'(t) + \kappa(t) \tau(t) \langle B(t), \alpha(t) - p \rangle = 0.
\]
Next note that, since $\{T, N, B\}$ is orthonormal,

\[
\begin{align*}
    r^2 &= \|\alpha(t) - p\|^2 \\
    &= \langle\alpha(t) - p, T(t)\rangle^2 + \langle\alpha(t) - p, N(t)\rangle^2 + \langle\alpha(t) - p, B(t)\rangle^2 \\
    &= 0 + \frac{1}{\kappa^2(t)} + \langle\alpha(t) - p, B(t)\rangle^2.
\end{align*}
\]

Thus, combining the previous two calculations, we obtain:

\[
-\frac{\kappa'(t)}{\kappa(t)} + \kappa(t)\tau(t)\sqrt{r^2 - \frac{1}{\kappa^2(t)}} = 0.
\]

**Exercise 7.** Check the converse, that is supposing that the curvature and torsion of some curve satisfies the above expression, verify whether the curve has to lie on a sphere of radius $r$.

To do the above exercise, we need to first find out where the center $p$ of the sphere could lie. To this end we start by writing

\[
p = \alpha(t) + a(t)T(t) + b(t)N(t) + c(t)B(t),
\]

and try to find $a(t), b(t)$ and $c(t)$ so that $p' = (0, 0, 0)$, and $\|\alpha(t) - p\| = r$. To make things easier, we may note that $\alpha(t) = 0$ (why?). Then we just need to find $b(t)$ and $c(t)$ subject to the two constraints mentioned above. We need to verify whether this is possible, when $\kappa$ and $\tau$ satisfy the above expression.