2.2 Definition of Gaussian Curvature

Let $M \subset \mathbb{R}^3$ be a regular embedded surface, as we defined in the previous lecture, and let $p \in M$. By the tangent space of $M$ at $p$, denoted by $T_pM$, we mean the set of all vectors $v$ in $\mathbb{R}^3$ such that for each vector $v$ there exists a smooth curve $\gamma: (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$ and $\gamma'(0) = v$.

**Exercise 1.** Let $H \subset \mathbb{R}^3$ be a plane. Show that, for all $p \in H$, $T_pH$ is the plane parallel to $H$ which passes through the origin.

**Exercise 2.** Prove that, for all $p \in M$, $T_pM$ is a 2-dimensional vector subspace of $\mathbb{R}^3$ (Hint: Let $(U, X)$ be a proper regular patch centered at $p$, i.e., $X(0,0) = p$. Recall that $dX_{(0,0)}$ is a linear map and has rank 2. Thus it suffices to show that $T_pM = dX_{(0,0)}(\mathbb{R}^2)$).

**Exercise 3.** Prove that $D_1X(0,0)$ and $D_2X(0,0)$ form a basis for $T_pM$ (Hint: Show that $D_1X(0,0) = dX_{(0,0)}(1,0)$ and $D_2X(0,0) = dX_{(0,0)}(0,1)$).

By a local gauss map of $M$ centered at $p$ we mean a pair $(V, n)$ where $V$ is an open neighborhood of $p$ in $M$ and $n: V \to \mathbb{S}^2$ is a continuous mapping such that $n(p)$ is orthogonal to $T_pM$ for all $p \in M$. For a more explicit formulation, let $(U, X)$ be a proper regular patch centered at $p$, and define $N: U \to \mathbb{S}^2$ by

$$N(u_1, u_2) := \frac{D_1X(u_1, u_2) \times D_2X(u_1, u_2)}{\|D_1X(u_1, u_2) \times D_2X(u_1, u_2)\|}.$$ 

Set $V := X(U)$, and recall that, since $(U, X)$ is proper, $V$ is open in $M$. Now define $n: V \to \mathbb{S}^2$ by

$$n(p) := N \circ X^{-1}(p).$$

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Exercise 4. Check that \((V, n)\) is indeed a local gauss map.

Exercise 5. Show that \(n : \mathbb{S}^2 \to \mathbb{S}^2\), given by \(n(p) := p\) is a Gauss map (Hint: Define \(f : \mathbb{R}^3 \to \mathbb{R}\) by \(f(p) := \|p\|^2\) and compute its gradient. Note that \(\mathbb{S}^2\) is a level set of \(f\). Thus the gradient of \(f\) at \(p\) must be orthogonal to \(\mathbb{S}^2\)).

Let \(M_1\) and \(M_2\) be regular embedded surfaces in \(\mathbb{R}^3\) and \(f : M_1 \to M_2\) be a smooth map (recall that this means that \(f\) may be extended to a smooth map in an open neighborhood of \(M_1\) in \(\mathbb{R}^3\)). Then for every \(p \in M_1\), we define a mapping \(df_p : T_pM_1 \to T_{f(p)}M_2\), known as the differential of \(M_1\) at \(p\) as follows. Let \(v \in T_pM_1\) and let \(\gamma_v : (-\epsilon, \epsilon) \to M_1\) be a curve such that \(\gamma(0) = p\) and \(\gamma'(0) = v\). Then we set
\[ df_p(v) := (f \circ \gamma_v)'(0). \]

Exercise 6. Prove that \(df_p\) is well defined (i.e. is independent of the smooth extension) and linear (Hint: Let \(\hat{f}\) be a smooth extension of \(f\) to an open neighborhood of \(M\). Then \(d\hat{f}_p\) is well defined. Show that for all \(v \in T_pM\), \(df_p(v) = d\hat{f}_p(v)\)).

Let \((V, n)\) be a local gauss map centered at \(p \in M\). Then the shape operator of \(M\) at \(p\) with respect to \(n\) is defined as
\[ S_p := -dn_p. \]

Note that the shape operator is determined up to two choices depending on the local gauss map, i.e., replacing \(n\) by \(-n\) switches the sign of the shape operator.

Exercise 7. Show that \(S_p\) may be viewed as a linear operator on \(T_pM\) (Hint: By definition, \(S_p\) is a linear map from \(T_pM\) to \(T_{n(p)}\mathbb{S}^2\). Thus it suffices to show that \(T_pM\) and \(T_{n(p)}\mathbb{S}^2\) coincide).

Exercise 8. A subset \(V\) of \(M\) is said to be connected if any pairs of points \(p\) and \(q\) in \(V\) may be joined by a curve in \(V\). Suppose that \(V\) is a connected open subset of \(M\), and, furthermore, suppose that the shape operator vanishes throughout \(V\), i.e., for every \(p \in M\) and \(v \in T_pM\), \(S_p(v) = 0\). Show then that \(V\) must be flat, i.e., it is a part of a plane (Hint: It is enough to show that the gauss map is constant on \(V\); or, equivalently, \(n(p) = n(q)\) for all
pairs of points $p$ and $q$ in $V$. Since $V$ is connected, there exists a curve 
$\gamma: [0, 1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$. Furthermore, since $V$ is open, we may choose $\gamma$ to be smooth as well. Define $f: [0, 1] \to \mathbb{R}$ by $f(t) := n \circ \gamma(t)$, and differentiate. Then $f'(t) = dn_{\gamma(t)}(\gamma'(t)) = 0$. Justify the last step and conclude that $n(p) = n(q)$.

**Exercise 9.** Compute the shape operator of a sphere of radius $r$ (Hint: Define $\pi: \mathbb{R}^3 - \{0\} \to S^2$ by $\pi(x) := x/\|x\|$. Note that $\pi$ is a smooth mapping and $\pi = n$ on $S^2$. Thus, for any $v \in T_p S^2$, $d\pi_p(v) = dn_p(v)$).

The **Gaussian curvature** of $M$ at $p$ is defined as the determinant of the shape operator:

$$K(p) := \det(S_p).$$

**Exercise 10.** Show that $K(p)$ does not depend on the choice of the local gauss map, i.e, replacing $n$ by $-n$ does not effect the value of $K(p)$.

**Exercise 11.** Compute the curvature of a sphere of radius $r$ (Hint: Use exercise 9).

Next we derive an explicit formula for $K(p)$ in terms of local coordinates. Let $(U, X)$ be a proper regular patch centered at $p$. For $1 \leq i, j \leq 2$, define the functions $g_{ij}: U \to \mathbb{R}$ by

$$g_{ij}(u_1, u_2) := \langle D_i X(u_1, u_2), D_j X(u_1, u_2) \rangle.$$

Note that $g_{12} = g_{21}$. Thus the above expression defines three functions. These are called the **coefficients of the first fundamental form** (a.k.a. *the metric tensor*) with respect to the given patch $(U, X)$. In the classical notation, these functions are denoted by $E$, $F$, and $G$ ($E := g_{11}$, $F := g_{12}$, and $G := g_{22}$). Next, define $l_{ij}: U \to \mathbb{R}$ by

$$l_{ij}(u_1, u_2) := \langle D_{ij} X(u_1, u_2), N(u_1, u_2) \rangle.$$

Thus $l_{ij}$ is a measure of the second derivatives of $X$ in a normal direction. $l_{ij}$ are known as the **coefficients of the second fundamental form** of $M$ with respect to the local patch $(U, X)$ (the classical notation for these functions are $L := l_{11}$, $M := l_{12}$, and $N := l_{22}$). We claim that

$$K(p) = \frac{\det(l_{ij}(0, 0))}{\det(g_{ij}(0, 0))}.$$
To see the above, recall that $e_i(p) := D_i X (X^{-1}(p))$ form a basis for $T_p M$. Thus, since $S_p$ is linear, $S_p(e_i) = \sum_{j=1}^2 S_{ij} e_j$. This yields that $\langle S_p(e_i), e_k \rangle = \sum_{j=1}^2 S_{ij} g_{jk}$. It can be shown that that $\langle S_p(e_i), e_k \rangle = l_{ik}$, see the exercise below. Then we have $[l_{ij}] = [S_{ij}] [g_{ij}]$, where the symbol $[\cdot]$ denotes the matrix with the given coefficients. Thus we can write $[S_{ij}] = [g_{ij}]^{-1} [l_{ij}]$ which yields the desired result.

**Exercise 12.** Show that $\langle S_p(e_i(p)), e_j(p) \rangle = l_{ij}(0,0)$ (*Hints:* First note that $\langle n(p), e_j(p) \rangle = 0$ for all $p \in V$. Let $\gamma: (-\epsilon, \epsilon) \to M$ be a curve with $\gamma(0) = p$ and $\gamma'(0) = e_i(p)$. Define $f: (-\epsilon, \epsilon) \to M$ by $f(t) := \langle n(\gamma(t)), e_j(\gamma(t)) \rangle$, and compute $f'(0)$.)

**Exercise 13.** Compute the Gaussian curvature of a surface of revolution, i.e., the surface covered by the patch

$$X(t, \theta) = (x(t) \cos \theta, x(t) \sin \theta, y(t)).$$

Next, letting

$$(x(t), y(t)) = (R + r \cos t, r \sin t),$$

i.e., a circle of radius $r$ centered at $(R, 0)$, compute the curvature of a torus of revolution. Sketch the torus and indicate the regions where the curvature is positive, negative, or zero.

**Exercise 14.** Let $(U, X)$ be a Monge patch, i.e,

$$X(u_1, u_2) := (u_1, u_2, f(u_1, u_2)),$$

centered at $p \in M$. Show that

$$K(p) := \frac{\det (Hess f(0,0))}{(1 + \| \text{grad } f(0,0) \|^2)^2},$$

where $Hess f := [D_{ij} f]$ is the Hessian matrix of $f$ and $\text{grad } f$ is its gradient.

**Exercise 15.** Compute the curvature of the graph of $z = ax^2 + by^2$, where $a$ and $b$ are constants. Note how the signs of $a$ and $b$ effect the curvature and shape of the surface. Also note the values of $a$ and $b$ for which the curvature is zero.